DAMAGE BOOK

PREFACE

Physics by Haliday and Resnick has been in use for numerous undergraduate and engineering courses all over the world for over a quarter of a century. The book is so ambitious that for want of space a sizable amount of material, which, in the traditional approach, is normally worked out in the text, appears at the end of each chapter as questions and problems and it is expected that the teachers and students would work them out as supplementary material for the text. Of the numerous merits which the textbook enjoys, probably the selection of problems is the most outstanding. The problems have been selected with the purpose of illustrating the underlying physical principles and have a variety which ranges from the "plugin" type to the sophisticated bordering on "brain-teasers."

Furthermore, the textbook is a rich source of problems and is ideally suited for setting examination papers at various levels. Students must, therefore, get acquainted with the techniques for solving the problems. To this end these solutions to all the problems, (about 1450 in number) from the *Physics*, of Resnick and Halliday and Halliday and Resnick, Parts I and II, respectively have been prepared. While detailed solutions have been provided for most of the problems, for a few alternative solutions have also been given.

An attempt has been made to retain the terminology of the textbook as the solutions are likely to be used by readers who are already in its possession. Solutions are given in the same units as in the problems. As the textbook does demand a prerequisite in calculus, problems have been freely using calculus methods. A few problems have warranted the use of non-calculus and the alternative solutions have been given. Solutions to additional *supplementary problems* given at the end of Part II have been presented at end of the corresponding chapters.

We hope that the *Solutions* will meet the requirements of both the teachers and the students.

AHMED ANWAR KAMAL

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CONTENTS

26 CHARGE AND MATTER

26.1. Number of protons/meter^2-sec falling over earth's surface,

$$n = 0.15 \times 10^4 = 1500$$

Number of protons/sec received by the entire earth's surface

$$N = 4\pi R^2 \, n$$

Where R is earth's radius.

$$N = (4\pi)\,(6.4 \times 10^6 \text{ meter})^2\,(1500) \text{ per sec}$$
$$= 7.717 \times 10^{17} \text{ per sec}$$

Since each proton carries charge $q = 1.6 \times 10^{-19}$ coulomb, total current received by earth

$$i = Nq = (7.717 \times 10^{17})\,(1.6 \times 10^{-19}) \text{ amp}$$
$$= 0.123 \text{ amp.}$$

26.2. The mangitude of the force on each charge is

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}$$

$$= \frac{(3 \times 10^{-6} \text{ coul})(1.5 \times 10^{-6} \text{ coul})}{(0.12 \text{ meter})^2}\,(9 \times 10^9 \text{nt-m}^2/\text{coul}^2)$$

$$= 2.8 \text{ nt}$$

The force is attractive.

26.3. Consider one of the balls which is in equilibrium under the joint action of three forces, the weight of the ball mg, the coulomb's repulsive force F, and the tension in the string T. (Fig. 26.3)

Balancing the vertical component of forces

$$T \cos \theta = mg \qquad \ldots(1)$$

Balancing the horizontal component of forces

$$T \sin \theta = F \qquad \ldots(2)$$

Dividing (2) by (1)

$$\tan \theta = F/mg$$

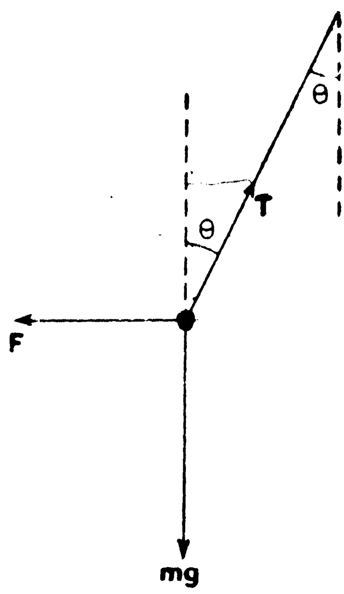


Fig. 26.3

$$F = mg \tan\theta \simeq mg \sin\theta$$

Tan θ = Sin θ − θ ∵ θ is very small

$$F = \frac{mgx}{2l}$$

But
$$F = \frac{q^2}{4\pi\varepsilon_0 x^2} = \frac{mgx}{2l}$$

or
$$x^3 = \frac{q^2 l}{2\pi\varepsilon_0 mg}$$

whence
$$x = \left(\frac{q^2 l}{2\pi\varepsilon_0 mg}\right)^{1/3}$$

That is,
$$q = \pm\sqrt{\frac{2\pi\varepsilon_0 mgx^3}{l}}$$

$$= \pm\sqrt{\frac{(2\pi)(8.9\times10^{-12}\text{ coul}^2/\text{nt-m}^2)(0.01\text{ kg})(9.8\text{m/s}^2)(0.05\text{m})^3}{1.2\text{ meter}}}$$

$$= \pm 2.39\times10^{-8}\text{ coulomb.}$$

26.4. $x = \left(\frac{q^2 l}{2\pi\varepsilon_0 mg}\right)^{1/3}$

Initial relative speed,
$$v = \frac{dx}{dt} = \frac{dx}{dq}\cdot\frac{dq}{dt}$$

$$= \frac{2}{3q^{1/3}}\left(\frac{l}{2\pi\varepsilon_0 mg}\right)^{1/3}\frac{dq}{dt}$$

$$= \frac{2}{3q}\left(\frac{lq^2}{2\pi\varepsilon_0 mg}\right)^{1/3}\frac{dq}{dt} = \frac{2x}{3q}\cdot\frac{dq}{dt}$$

$$= \frac{2}{3}\,\frac{(50\ mm)(1.0\times10^{-9}\text{ coul/sec})}{(2.4\times10^{-8}\text{ coul})}$$

$$= 1.4\text{ mm/sec.}$$

26.5 The repulsive force F_{BA} on A due to charge B acts in the direction BA and is represented by AD. Similarly, the repulsive force F_{CA} on A due to charge C acts in the direction CA and is represented by AE. Resolve the forces $\mathbf{F}_{BA}$ and $\mathbf{F}_{CA}$ along two mutually perpendicular directions BC and ML. It is seen that along BC the components of $\mathbf{F}_{BA}$ and $\mathbf{F}_{CA}$ being equal in magnitude but

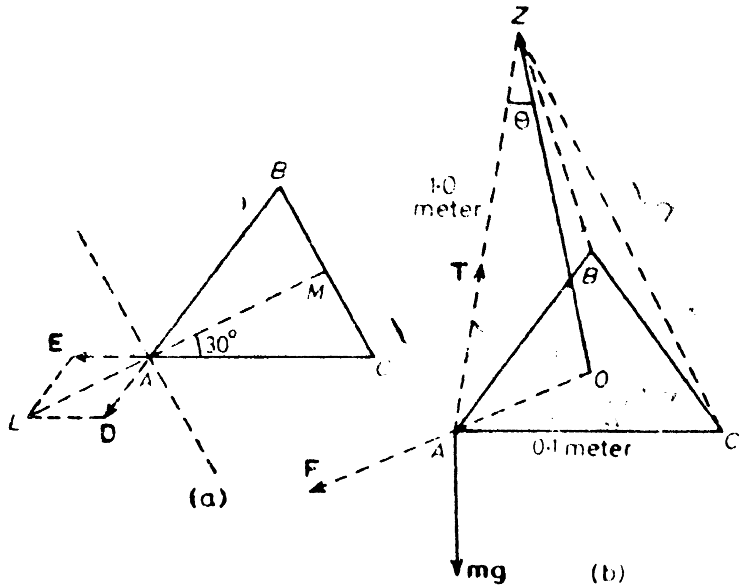


Fig. 26.5

opposite in direction get cancelled out. On the other hand, the components along ML get added up. Thus the net force on A is given by

$$F = F_{BA} \cos 30° + F_{CA} \cos 30° \qquad \ldots(1)$$

but
$$F_{BA} = F_{CA} = \frac{q^2}{4\pi\varepsilon_0 x^2} \qquad \ldots(2)$$

where $x = 0.1$ meter.

Consider the ball A which is in equilibrium under the joint action of three forces, repulsive force F due to B and C, tension of the thread T and the weight of the ball mg. These three forces are in the same plane.

Balancing the vertical components

$$T \cos \theta = mg \qquad \ldots(3)$$

where θ is the angle made by the thread with the vertical.

Balancing the horizontal components

$$T \sin \theta = F \qquad \ldots(4)$$

Divide (4) by (3)

$$\tan\theta = F/mg$$

$$F = mg\tan\theta \qquad ...(5)$$

From the geometry of the figure we find $OA = \frac{0.1}{\sqrt{3}}$ meter.

$$\therefore \qquad \sin\theta = (0.1/\sqrt{3})/1.0 = \frac{0.1}{\sqrt{3}} \qquad ...(6)$$

Since θ is small,

$$\tan\theta \simeq \sin\theta \qquad ...(7)$$

Combining (1), (2), (5) (6) and (7)

$$F = mg\tan\theta = mg\sin\theta = 2F_{BA}\cos 30° = \sqrt{3}\,F_{BA}$$

$$= \frac{\sqrt{3}\,q^2}{4\pi\varepsilon_0 X^2}$$

$$\text{or}\quad q^2 = \frac{4\pi\varepsilon_0 x^2\,mg\sin\theta}{\sqrt{3}} = \frac{(0.1\text{ meter})^2\,(0.01\text{ kg})\,(9.8\text{ m/s}^2)}{(9\times10^9\text{ nt-m}^2/\text{coul}^2)}\,\frac{0.1}{\sqrt{3}}$$

$$= 36.3\times10^{-16}\text{ coul}^2$$

$$\therefore \qquad q = 6\times10^{-8}\text{ coul.}$$

26.6. Choose the origin at the charge 1 in the lower left corner of the square. The force F_{12} due to charge 2 on 1 has components

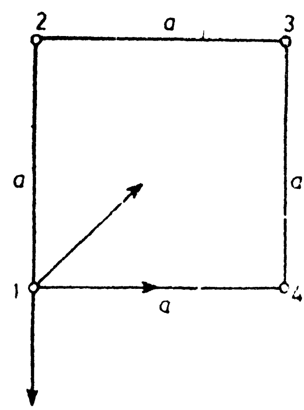


Fig. 26.6

$$F_x(12) = 0$$

$$F_y(12) = F_{12}$$

$$= -\frac{(q)(2q)}{4\pi\varepsilon_0 a^2}$$

$$= -\frac{q^2}{2\pi\varepsilon_0 a^2}$$

The force F_{13} due to charge 3 on 1 has components

$$F_x(13) = F_{13}\cos 45° = \frac{(q)\,(2q)\cos 45°}{4\pi\varepsilon_0\,(\sqrt{2}\,a)^2} = \frac{q^2}{4\pi\varepsilon_0\sqrt{2}\,a^2}$$

$$F_y(13) = F_{13}\sin 45°$$

$$= \frac{(q)\,(2q)\sin 45°}{4\pi\varepsilon_0\,(\sqrt{2}\,a)^2} = \frac{q^2}{4\pi\varepsilon_0\sqrt{2}\,a^2}$$

The force F_{14} due to charge 4 on 1 has components

$$F_x(14) = F_{14} = \frac{(2q)\,(2q)}{4\pi\varepsilon_0 a^2} = \frac{q^2}{\pi\varepsilon_0\,a^2}$$

$$F_y(14) = 0$$

Net x-component of force is given by

$$F_x = F_x\,(12) + F_x\,(13) + F_x\,(14) = 0 + \frac{q^2}{4\pi\varepsilon_0\sqrt{2}\,a^2} + \frac{q^2}{\pi\varepsilon_0\,a^2}$$

$$= \frac{q^2\,(1+4\sqrt{2})}{4\pi\varepsilon_0\sqrt{2}\,a^2}$$

$$= \frac{(1.0\times10^{-7}\text{ coul})^2\,(9\times10^9\text{ nt-m}^2/\text{coul}^2)\,(1+4\sqrt{2})}{\sqrt{2}\,(0.05\text{ meter})^2}$$

$$= 0.17\text{ nt}$$

Net y-component of force is given by

$$F_y = F_y(12) + F_y(13) + F_y(14)$$

$$= -\frac{q^2}{2\pi\varepsilon_0 a^2} + \frac{q^2}{4\pi\varepsilon_0\sqrt{2}a^2} + 0$$

$$= -\frac{q^2(2\sqrt{2}-1)}{4\pi\varepsilon_0\sqrt{2}\,a^2}$$

$$= -\frac{(1.0\times10^{-7}\text{ coul})^2(9\times10^9\text{ nt-m/coul}^2)(2\sqrt{2}-1)}{\sqrt{2}\,(0.05\text{ meter})^2}$$

$$= -0.046\text{ nt}$$

26.7. (a) The force due to Q_2 on Q_1 is repulsive and will be directed along the diagonal Q_2Q_1 of the square of side a.

$$F_{QQ} = \frac{Q^2}{4\pi\varepsilon_0(\sqrt{2}\,a)^2}$$

$$= \frac{Q^2}{8\pi\varepsilon_0\,a^2}$$

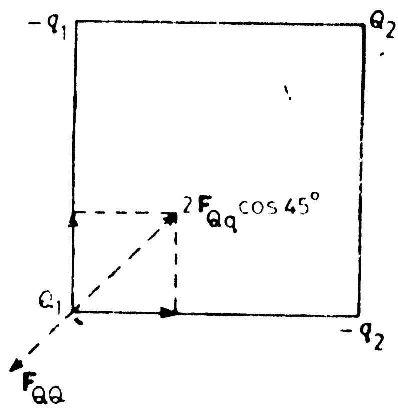


Fig. 26.7

The forces due to $-q_1$ and $-q_2$ on Q_1 are attractive and are directed along the sides of the square as shown in Fig 26.7. Resolve the forces due to the changes $-q_1$ and $-q_2$ along the diagonal Q_1Q_2 and perpendicular to it. Along the perpendicular direction the components get cancelled. On the other hand along the diagonal QQ the components get added up.

Due to two charges of magnitude q,

$$2F_{Qq}\cos 45^\circ = -\frac{2Qq}{4\pi\varepsilon_0\sqrt{2}\,a^2}$$

If the resultant electric force on Q is zero, then we must have

$$F_{QQ} = 2\,F_{Qq}\cos 45^\circ$$

$$\frac{Q^2}{8\pi\varepsilon_0\,a^2} = -\frac{2Qq}{4\pi\varepsilon_0\sqrt{2}\,a^2}$$

whence $$Q=-2\sqrt{2}\,q \qquad \ldots(1)$$

(*b*) If the resultant force on q is to be zero, then the condition would be

$$q=-2\sqrt{2}\,Q \qquad \ldots(2)$$

Obviously both (1) and (2) cannot be satisfied simultaneously. Therefore, no matter how q is chosen the resultant force on every charge cannot be zero.

26.8. Let the protons each of charge q coul, be at distance d apart. Then the electrical repulsive force on either one is

$$F=\frac{q^2}{4\pi\varepsilon_0 d^2}=mg$$

by Problem,

$$\therefore \qquad d=\frac{q}{\sqrt{4\pi\varepsilon_0 mg}}=(1.6\times10^{-19}\text{ coul})$$

That is, $$\sqrt{\frac{9\times10^9\text{ nt-m}^2/\text{coul}^2}{(1.66\times10^{-27}\text{ kg})(9.8\text{ meter/sec}^2)}}$$

$$=0.119\text{ meter}=11.9\text{ cm}$$

26.9. (*a*) Let a charge $+Q$ be placed on earth and an equal amount on moon. By Problem, electrical repulsive force $=$ gravitational attractive force.

$$\frac{Q^2}{4\pi\varepsilon_0 d^2}=\frac{GMm}{d^2}$$

where M and m are the masses of earth and moon respectively and d is the distance of separation.

$$Q=\sqrt{4\pi\varepsilon_0\,GMm}$$

$$=\sqrt{\frac{(6.7\times10^{-11}\text{ nt-m}^2/\text{kg}^2)(6\times10^{24}\text{ kg})(7.37\times10^{22}\text{ kg})}{(9\times10^9\text{ nt-m}^2/\text{coul}^2)}}$$

$$=5.74\times10^{13}\text{ coul.}$$

(*b*) No, since the distance d gets cancelled.

(*c*) Number of protons N each of charge q required to produce the charge Q is

$$N=\frac{Q}{q}=\frac{5.74\times10^{13}\text{ coul}}{1.6\times10^{-19}\text{ coul}}=3.6\times10^{32}$$

Neglecting the mass of electron, mass of hydrogen required

$$=\frac{N}{N_0}=\frac{3.6\times10^{32}}{6\times10^{23}}=6\times10^{8}\text{ gm}$$

$$=(6\times10^{8}\text{ gm})(1.1\times10^{-6}\text{ ton/gm})$$

$$=660\text{ ton}$$

26.10. Coulomb force between two parts q and $Q-q$ placed at a distance d apart is

$$F=\frac{q(Q-q)}{4\pi\varepsilon_0\, d^2}$$

Holding Q and d as constant differentiate F with respect to q to get

$$\frac{\partial F}{\partial q}=\frac{Q-2q}{4\pi\varepsilon_0\, d^2}$$

For maximum force, set

$$\frac{\partial F}{\partial q}=0$$

$$Q-2q=0$$

That is, $$q=\frac{Q}{2}$$

Thus, the charge Q must be divided equally in order to get maximum repulsion. We can test whether it is actually a maximum by finding the sign of $\frac{\partial^2 F}{\partial q^2}$. We get, $\frac{\partial^2 F}{\partial q^2}=-\frac{1}{2\pi\varepsilon_0 d^2}$. Since the sign is negative it is a maximum.

26.11. Let the charges be q and $(5\times10^{-5}-q)$ coul.

$$F=\frac{q_1q_2}{4\pi\varepsilon_0 d^2}=\frac{q(5\times10^{-5}-q)(9\times10^9\ \text{nt-m}^2/\text{coul}^2)}{(2.0\ \text{meter})^2}=1.0\ \text{nt}$$

Simplifying,

$$9\times10^9\, q^2-45\times10^4 q+4=0$$

Solution of the quadratic equation yields

$$q=1.2\times10^{-5}\ \text{coul and}\ 3.8\times10^{-5}\ \text{coul}.$$

26.12. Let the test charge $+q$ be placed at C a distance r on the bisector of the line AB joining Q_1 and Q_2. The force F_{Q_1q} of Q_1 on q is directed along AC and is represented by CF. Similarly, the force F_{Q_2q} due to Q_2 on q is directed along BC and is represented by CD. Since $Q_1=Q_2=Q$ and the sides $AC=BC$, these forces are equal and consequently $CD=CF$. Complete the parallelogram $CDEF$ which is actually a rhombus. The resultant is given by the diagonal CE which lies on the bisector of AB.

$$F_{Q_1q}=\frac{Qq}{4\pi\varepsilon_0(AC)^2}=\frac{Qq}{4\pi\varepsilon_0(r^2+a^2)}$$

$$F_{Q_2q}=\frac{Qq}{4\pi\varepsilon_0(BC)^2}=\frac{Qq}{4\pi\varepsilon_0(r^2+a^2)}$$

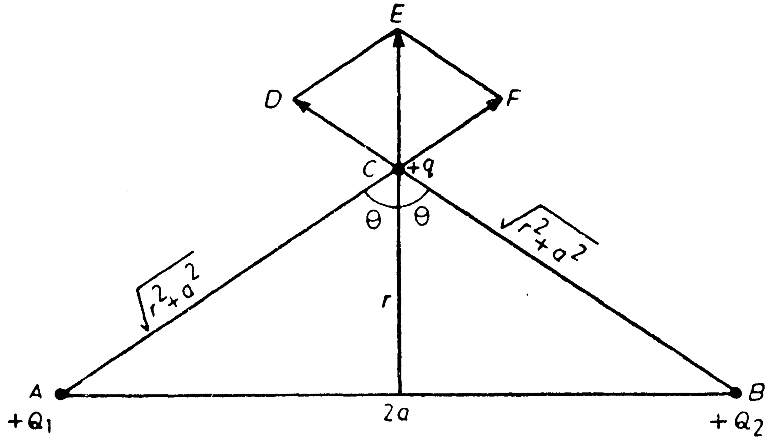


Fig. 26.12

Resultant force on q is given by

$$F = F_{Q_1q}\cos\theta + F_{Q_2q}\cos\theta = \frac{Qq\cos\theta}{2\pi\varepsilon_0(r^2+a^2)}$$

But
$$\cos\theta = \frac{r}{\sqrt{r^2+a^2}}$$

$\therefore$
$$F = \frac{Qqr}{2\pi\varepsilon_0(r^2+a^2)^{3/2}}$$

For F to be maximum, set $\dfrac{\partial F}{\partial r}=0$

$$\frac{Qq[(r^2+a^2)^{3/2} - 3\,r^2(r^2+a^2)^{\frac{1}{2}}]}{2\pi\varepsilon_0\,(r^2+a^2)^3} = 0$$

whence $(r^2+a^2)^{3/2} - 3r^2(r^2+a^2)^{\,1/2} = 0$

or
$$r^2+a^2-3r^2=0$$

$$r = \frac{a}{\sqrt{2}}$$

(*b*) The direction of force is along the bisector of the line joining the two original charges and away from the line.

26.13. (*a*) Let us first calculate the x-component on charge 1 due to charges 2, 3, 4, 5, 6, 7 and 8. Choose origin at charge 1.

$$F_{x_{12}} = 0$$
$$F_{x_{13}} = F_{13}\cos 45^\circ$$
$$= \frac{q^2}{4\pi\varepsilon_0(\sqrt{2}\,a)^2}\cdot\frac{1}{\sqrt{2}}$$
$$= \frac{q^2}{8\sqrt{2}\pi\varepsilon_0\,a^2}$$

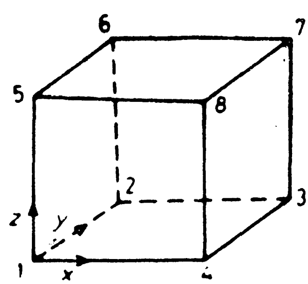


Fig. 26.13

$$F_{x14}=F_{14}=\frac{q^2}{4\pi\varepsilon_o a^2}$$

$$F_{x15}=0$$

$$F_{x16}=0$$

$$F_{x17}=\frac{F_{17}}{\sqrt{3}}=\frac{q^2}{4\pi\varepsilon_o(\sqrt{3}\,a)^2}\cdot\frac{1}{\sqrt{3}}$$

$$=\frac{q^2}{12\sqrt{3}\pi\varepsilon_o a^2}$$

where we have used the fact that the direction cosine of the body diagonal with x-axis is $\frac{1}{\sqrt{3}}$.

$$F_{x18}=F_{18}\cos 45°=\frac{q^2}{4\pi\varepsilon_o(\sqrt{2}\,a)^2}\cdot\frac{1}{\sqrt{2}}$$

$$=\frac{q^2}{8\sqrt{2}\pi\varepsilon_o a^2}$$

The x-component of net force is then

$$F_x=\frac{q^2}{4\pi\varepsilon_o a^2}\left[1+\frac{1}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}\right]$$

$$=0.1512\,\frac{q^2}{\varepsilon_o a^2}$$

By symmetry F_y and F_z also have the same magnitude.

$$F=\sqrt{F_x^2+F_y^2+F_z^2}=\sqrt{3}\,F_x$$

$$=(1.732)(0.1512)\,\frac{q^2}{\varepsilon_o a^2}=0.262\,\frac{q^2}{\varepsilon_o a^2}$$

(b) The force is directed along the body diagonal.

26.14. Assuming 500 cm^3 of water, (m=500 gm), number of water molecules.

$$N=\frac{N_o m}{M}$$

where N_o is the Avogadro's number and M is the molecular weight.

$$N=\frac{(6.03\times10^{23})(500\text{ gm})}{(18\text{ gm})}$$

$$=1.67\times10^{25}\text{ molecules.}$$

Since each water molecule has 10 protons, number of protons in the sample$=1.67\times10^{25}\times10=1.67\times10^{26}$.

Now each proton carries $+1.6\times10^{-19}$ coul charge.

Hence, positive charge in the glass of water

$$=(1.6\times10^{-19}\text{ coul})\,(1.67\times10^{26})$$
$$=2.7\times10^{7}\text{ coul.}$$

26.15. (*a*) Since the penny is originally neutral, the charge Q associated with n, the number of electrons, that are to be removed is equal to -10^{-7} coul.

If q is the charge carried by each electron, then

$$n=\frac{Q}{q}=\frac{-10^{-7}\text{ coul}}{-1.6\times10^{-19}\text{ coul}}=6.25\times10^{11}$$

(b) The number N of copper atoms in a penny is found from

$$N=N_0\,\frac{m}{A}$$

where N_0 is the Avogadro's number, m the mass of the coin and A the atomic weight of copper. Assuming that $m=3.1$ gm,

$$N=\frac{(6.03\times10^{23}\text{ atoms/mole})\,(3.1\text{ gm})}{64\text{ gm/mole}}$$
$$=2.9\times10^{22}\text{ atoms.}$$

As there are 29 electrons in each atom of copper, total number of electrons in the penny is given by

$$n_0=29N=(29)\,(2.9\times10^{22})=8.41\times10^{23}\text{ electrons}$$

$$\text{The fraction } f=\frac{n}{n_0}=\frac{6.25\times10^{11}}{8.41\times10^{23}}=7.4\times10^{-13}$$

26.16 Mass of copper atom $M=\dfrac{A}{N_0}$

where A is the atomic weight of copper and N_0 is Avogadro's number

$$M=(64\text{ gm/mole})/(6.03\times10^{23}\text{ atoms/mole})$$
$$=1.06\times10^{-22}\text{ gm}$$
$$=1.06\times10^{-25}\text{ kg}$$

Volume of copper nucleus,

$$v=\frac{4}{3}\pi R^3=\frac{4}{3}\pi(1.9\times10^{-15}\text{ meter})^3$$
$$=2.87\times10^{-44}\text{ meter}^3$$

As mass of the nucleus is approximately that of the atom,

$$\text{Nuclear density, } \frac{M}{v} = (1.06 \times 10^{-25} \text{ kg})/(2.87 \times 10^{-44} \text{ meter}^3)$$

$$= 3.7 \times 10^{19} \text{ kg/meter}^3.$$

The answer is reasonable since ordinary density of materials is of the order of 10^4 kg/meter^3, and because nuclei have radii which are smaller by a factor of 10^5 compared to atomic radii, the nuclear volume is smaller by a factor of $(10^5)^3$ or 10^{15}, and nuclear densities are expected to be larger by a similar factor *i.e.*, would be of the order of (10^{15}) $(10^4$ kg/meter$^3)$ or 10^{19} kg/meter^3 which is of the right order of magnitude.

26.17. (*a*) The coulomb force is given by

$$F = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 r^2}$$

where Z_1 and Z_2 are the atomic numbers of α-particle and Thorium nucleus and r is the distance of separation.

$$F = \frac{(2)\,(90)\,(1.6 \times 10^{-19} \text{ coul})^2 (9 \times 10^9 \text{ nt-m}^2/\text{coul}^2)}{(9 \times 10^{-15} \text{ meter})^2}$$

$$= 512 \text{ nt}$$

(*b*) Mass of α-particle $M = 6.69 \times 10^{-27}$ kg

$$\text{Acceleration, } a = \frac{F}{M} = \frac{512 \text{ nt}}{6.69 \times 10^{-27} \text{ kg}} = 7.7 \times 10^{28} \text{ meter/sec}^2$$

SUPPLEMENTARY PROBLEMS

S.26.1. For equilibrium, it is necessary that the third charge Q be placed on the line joining the other two charges. Let Q be located

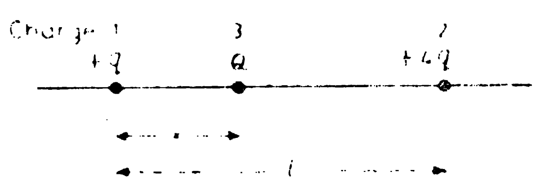


Fig. 26.14

at distance x from $+q$. The electric force F_{13} on Q due to $+q$ has magnitude.

$$F_{13} = \frac{1}{4\pi\varepsilon_0} \frac{qQ}{x^2}$$

The electric force F_{23} on Q due to charge $+4q$ has magnitude

$$F_{23} = \frac{1}{4\pi\varepsilon_0} \frac{4qQ}{(l-x)^2}$$

Condition for equilibrium of Q is

$$F_{13}=F_{23}$$

i.e. $$\frac{1}{4\pi\varepsilon_0}\frac{qQ}{x^2}=\frac{1}{4\pi\varepsilon_0}\frac{4q\,Q}{(l-x)^2}$$

After cancelling the obvious common terms,

$$\frac{1}{x^2}=\frac{4}{(1-x)^2}$$

which yields the solution $x=l/3$

Next, consider the equilibrium of $+q$. The force F_{21} due to charge $+4q$ *on* $+q$ has magnitude.

$$F_{21}=\frac{1}{4\pi\varepsilon_0}\frac{(4q)(q)}{l^2}$$

The force F_{31} due to Q on $+q$ has magnitude

$$F_{31}=\frac{1}{4\pi\varepsilon_0}\frac{Qq}{x^2}$$

As the force due to $+4q$ on $+q$ is repulsive, that due to Q on $+q$ should be attractive so that $+q$ may be in equilibrium.

We must then have

$$F_{31}=-F_{21}$$

$$\frac{1}{4\pi\varepsilon_0}\frac{Qq}{x^2}=-\frac{1}{4\pi\varepsilon_0}\frac{4q^2}{l^2}$$

Putting $x=l/3$ and solving for Q, we find

$$Q=-\frac{4q}{9}$$

S.26.2 By Problem 26.3 the distance of separation between the charged balls is given by

$$x=\left[\frac{q^2 l}{2\pi\varepsilon_0 mg}\right]^{1/3} \quad \ldots(1)$$

For $l=120$ cm, $x=5$ cm, $m=10$ gm, we find $q=\pm 2.4\times 10^{-8}$ coul.

When one of the balls gets discharged they come into contact, share the charge of the other ball equally so that each of them will now carry charge $\frac{1}{2}q=+1.2\times 10^{-8}$ coul and will mutually repel each other. For the new charge $\frac{1}{2}q$, formula (1) gives the new distance x',

$$x'=\left[\frac{(\frac{1}{2}q)^2}{2\pi\varepsilon_0}\frac{l}{mg}\right]^{1/3} \quad \ldots(2)$$

Dividing (2) by (1),

$$\frac{x'}{x}=\frac{1}{(4)^{1/3}}=0.63$$

$$x'=0.63x=(0.63)\,(5\text{ cm})=3.15\text{ cm.}$$

S.26.3. Let initial charges on the spheres be Q and q. When they are separated by distance r they attract each other with a force

$$F = -\frac{Qq}{4\pi\varepsilon_o r^2}$$

$$Qq = -4\pi\varepsilon_o r^2 F = -\frac{(0.5 \text{ meter})^2 (0.108 \text{ nt})}{9\times 10^9 \text{ nt-m}^2/\text{coul}^2}$$

$$= -3\times 10^{-12} \text{ (coul)}^2 \qquad \ldots(1)$$

When the spheres are connected by a wire then the net charge $(Q-q)$ is shared equally by the two spheres since they are identical. Thus, each of them now carries charge $\frac{1}{2}(Q-q)$.

They now repel by a force given by

$$F' = [\tfrac{1}{2}(Q-q)]^2/4\pi\varepsilon_o r^2$$

$$\therefore\ (Q-q)^2 = 16\pi\varepsilon_o F' r^2 = \frac{(4)(0.036 \text{ nt})(0.5 \text{ meter})^2}{9\times 10^9 \text{ nt-m}^2/\text{coul}^2}$$

$$= 4\times 10^{-12} (\text{coul})^2$$

$$Q-q = \pm 2\times 10^{-6} \qquad \ldots(2)$$

Solving (1) and (2), $Q = \pm 3\times 10^{-6}$ coul.

$q = \mp 1\times 10^{-6}$ coul.

S.26.4. (*a*) The coulomb force is

$$Fc = \frac{1}{4\pi\varepsilon_o}\frac{Qq}{r^2} \qquad \ldots(1)$$

The centripetal force is

$$F = \frac{mv^2}{r} = m\omega^2 r \qquad \ldots(2)$$

Equating the coulomb force to the centripetal force,

$$\frac{1}{4\pi\varepsilon_o}\frac{Qq}{r^2} = m\omega^2 r = m\left(\frac{2\pi}{T}\right)^2 r$$

$$r^3 = \frac{QqT^2}{16\pi^3 \varepsilon_o m}$$

(*b*) The gravitational force between two particles of mass m and M is

$$F_{gr} = \frac{G\,mM}{r^2} \qquad (3)$$

The centripetal force is as given by (2).

Equating the gravitational force to the centripetal force,

$$\frac{GmM}{r^2} = m\omega^2 r = m\left(\frac{2\pi}{T}\right)^2 r$$

$$r^3 = \frac{GMT^2}{4\pi^2}.$$

S.26.5. Let the electron be projected with initial speed v_o at infinite distance from proton. Let it acquire a speed v at distance r from the proton. The gain in kinetic energy

$$\Delta K = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_o^2 \qquad \ldots(1)$$

Initially, the electron being at infinite distance has potential energy $U_o=0$. At distance r the magnitude of potential energy is

$$U = \frac{e^2}{4\pi\varepsilon_o r} \qquad \ldots(2)$$

Loss in potential energy is

$$\Delta U = U - U_o = \frac{e^2}{4\pi\varepsilon_o r} \qquad \ldots(3)$$

By work-energy priniciple, gain in kinetic energy is equal to loss in potential energy,

$$\Delta K = \Delta U \qquad \ldots(4)$$

Using (1) and (3) in (4),

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv^2_o = \frac{e^2}{4\pi\varepsilon_o r} \qquad \ldots(5)$$

By Problem, $\qquad v = 2v_o \qquad \ldots(6)$

Using (6) in (5) and solving for r,

$$r = \left(\frac{2}{3}\right)\frac{e^2}{4\pi\varepsilon_o\, mv_o^2}$$

$$= \left(\frac{2}{3}\right)\frac{(1.6\times10^{-19}\text{ coul})^2\,(9\times10^9\text{nt-m}^2/\text{coul}^2)}{(9.1\times10^{-31}\text{ kg})\,(3.24\times10^5\text{ meter/sec})^2}$$

$$= 1.61\times10^{-9}\text{ meter.}$$

S.26.6. (*a*) Force exerted on the left dipole

$$F = \frac{q^2}{4\pi\varepsilon_o}\left[-\frac{1}{R^2} + \frac{1}{(R-2a)^2} - \frac{1}{R^2} + \frac{1}{(R+2a)^2}\right]$$

$$= \frac{q^2}{4\pi\varepsilon_o}\left[\frac{2R^2+8a^2}{(R^2-4a^2)^2} - \frac{2}{R^2}\right]$$

$$= \frac{q^2}{2\pi\varepsilon_o}\left[\frac{R^2+4a^2}{(R^2-4a^2)^2} - \frac{1}{R^2}\right]$$

(*b*) The above result can be rewritten as

$$F = \frac{q^2}{2\pi\varepsilon_o}\left[\frac{12R^2a^2-16a^4}{(R^2-4a^2)^2R^2}\right]$$

Now, $R \gg a$ Neglecting the second term in the numerator in comparison with the first one, and approximating R^2-4a^2 by R^2 in the denominator,

$$F = \frac{q^2}{2\pi\varepsilon_o}\;\frac{12R^2a^2}{(R^2)^2R^2} = \frac{(3)\,(4q^2a^2)}{2\pi\varepsilon_o\, R^4} = \frac{3p^2}{2\pi\varepsilon_o\, R^4}$$

with $\qquad p = 2qa.$

27 THE ELECTRIC FIELD

27.1.

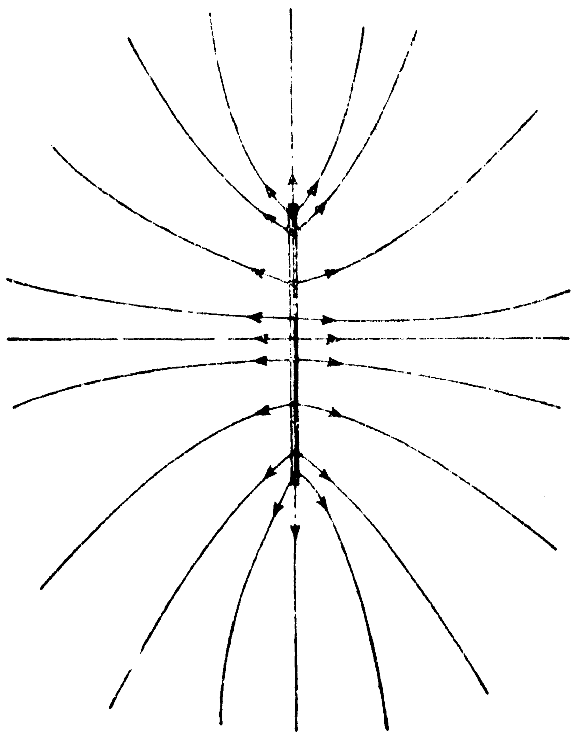


Fig. 27.1

27.2. (*a*) Lines of force due to equal charges placed at *A*, *B* and *C*, the vertices of equilateral triangle are shown in Fig. 27.2.

(*b*) The test charge is kept at *D*, the center of the triangle. Let $AD=a$. Take the origin at *D* and let the plane *ABC* be the *xy* plane. It can be shown that near *D* the potential is

$$V=\frac{3}{a}+\frac{3}{4a^3}(x^2+y^2-2z^2)$$

where all the charges are assumed to be equal to unity. It is easily deduced from the above formula that the potential at *D* is not a minimum for all directions in space. As we move away from *D* in directions lying in the plane *ABC*, the potential increases. on the other hand, the potential decreases as we move away in a direction

perpendicular to this plane. Thus, the test charge at D, although in stable equilibrium in the plane ABC, is in unstable equilibrium off this plane.

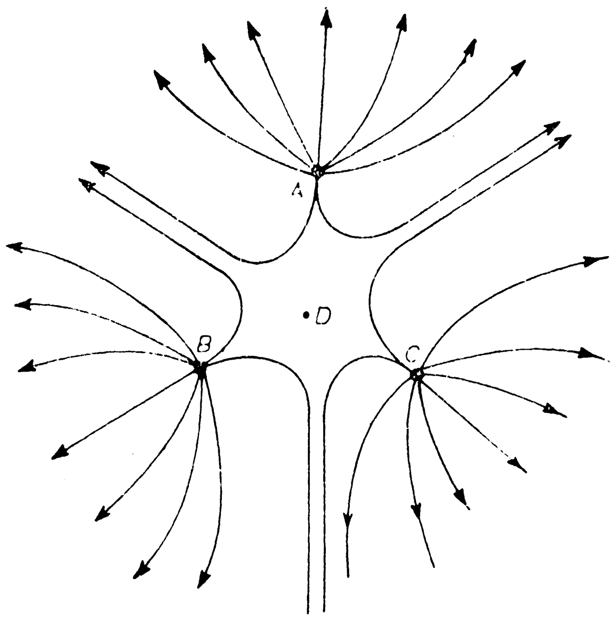


Fig. 27.2

27.3. From the upper charge, initially the lines of force tend to be projected radially outward with an angular seperation determined by the electric field strength. In the absence of the lower charge the same angular separation should have been maintained at a large distance. However, actually in the presence of the second charge which is equal in magnitude, the relative field strength at any distant point must be doubled as now the lines of force are arising from two charges rather than a single charge, thereby reducing the angular separation to half of its previous value. Hence, if the angle between the tangents to any two lines of force leaving the upper charge is θ, it becomes $\frac{1}{2}\theta$ at great distance.

27.4. Let the electrical field intensity at a distance r from a point charge q be given by

$$E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^n}$$

where $n \neq 2$; the field direction will be radially outward from the charge and for any point of a spherical surface of radius r concentric with the isolated point charge, the direction of the field will be perpendicular to the spherical surface. If A is the surface area then

the total number of lines threading through the sphere is therefore,

$$N = EA = \frac{(q)\,(4\pi r^2)}{4\pi\varepsilon_0 r^n} = \frac{q}{\varepsilon_0 r^{n-2}}$$

This shows that the number of lines is dependent on the radius of the sphere, from which it follows that the same number of lines do not cross every sphere concentric with the charge. This then means that some of the lines of force may originate or terminate in the space surrounding the sphere so that the assumed continuity of lines of force will be violated. (It is only in the case of $n=2$ that N is independent of r.)

27.5. Let the magnitude of the point charge chosen be q_0. The electric field

$$E = \frac{F}{q} = \frac{qq_0}{4\pi\varepsilon_0 r^2 q} = \frac{q_0}{4\pi\varepsilon_0 r^2}$$

$$q_0 = 4\pi\varepsilon_0 r^2 E = \frac{(0.5 \text{ meter})^2(2.0 \text{ nt/coul})}{(9\times 10^9 \text{nt-m}^2/\text{coul}^2)} = 5.5\times 10^{-11} \text{ coul.}$$

27.6. (*a*) Consider a positive test charge placed midway between the given charges. The test charge will be attracted by the negative charge and repelled by the positive one so that the electric field E will be directed towards the negative charge.

$$E = \frac{2q}{4\pi\varepsilon_0 r^2} = \frac{(2)(2\times 10^{-7} \text{ coul})(9\times 10^9 \text{nt-m}^2/\text{coul}^2)}{(0.075 \text{ meter})^2} \quad (0.15)$$

$= 6.4\times 10^5$ nt/coul towards the negative charge.

(*b*) Force on electron, $F = Ee = (6.4\times 10^5 \text{ nt/coul})(1.6\times 10^{-19} \text{ coul})$

$= 1.02\times 10^{-13}$ nt toward the +ve charge.

27.7. E is calculated from

$$E = \frac{q_1}{4\pi\varepsilon_0 x^2} + \frac{q_2}{4\pi\varepsilon_0 (x-d)^2}$$

Fig. 27.7 shows the plot of E versus x.

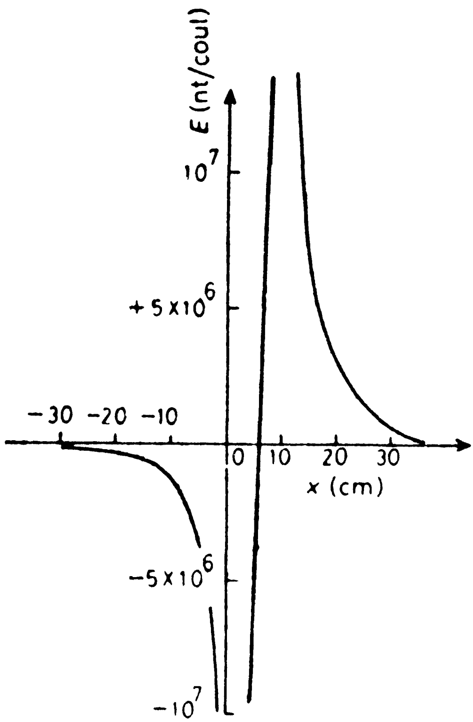


Fig. 27.7

27.8. The force F^+ due to $+Q$ on $+q$ and F^- due to $-Q$ on $+q$ are equal in magnitude, and are indicated in Fig. 27.8. The angle sub-

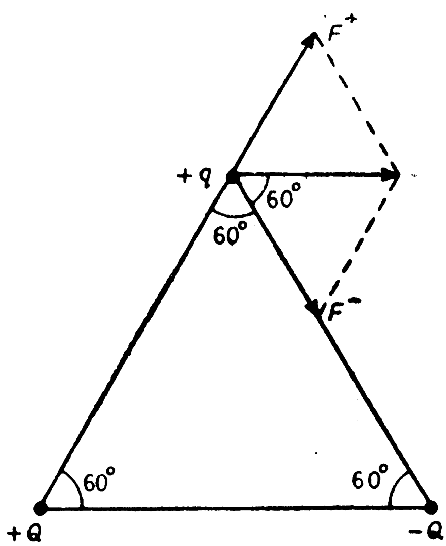


Fig. 27.8

tended between F^+ and F^- is 120°. The resultant force, F will make an angle 60° with F^-. Hence, the direction of force on $+q$ is parallel to the line joining $+Q$ and $-Q$.

27.9. (*b*) Let $q_1=+2.0\times10^{-7}$ coul and $q_2=+8.5\times10^{-8}$ coul. Force on each charge is given by

$$F=\frac{q_1q_2}{4\pi\varepsilon_0r^2}$$

$$=\frac{(9\times10^9\text{ nt-m}^2/\text{coul}^2)(2\times10^{-7}\text{ coul})(8.5\times10^{-8}\text{ coul})}{(0.12\text{ meter})^2}$$

$$=1.06\times10^{-2}\text{ nt}$$

(*a*) Electric field produced by q_1 at the site of q_2 is

$$E_1=\frac{F}{q_2}=\frac{1.06\times10^{-2}\text{ nt}}{8.5\times10^{-8}\text{ coul}}=1.25\times10^5\text{ nt/coul}$$

Electric field produced by q_2 at at the site of q_1 is

$$E_2=\frac{F}{q_1}=\frac{1.06\times10^{-2}\text{ nt}}{2\times10^{-7}\text{ coul}}=5.3\times10^4\text{ nt/coul.}$$

27.10. (*a*) Electric field at P is given by

$$E=E^+ + E^-$$

Where E^+ is the field due to charge ^+q and E^- is the field due to $-q$,

$$E=\frac{q}{4\pi\varepsilon_0(r-a^2)^2}-\frac{q}{4\pi\varepsilon_0\,(r+a)^2}$$

$$=\frac{qar}{\pi\varepsilon_0\,(r^2-a^2)^2}$$

If $r \gg a$, then a^2 can be neglected in comparison with r^2 in the denominator.

$$E=\frac{qa}{\pi\varepsilon_0\,r^3}$$

By definition, the dipole moment $p=2aq$.

$$\therefore \qquad E=\frac{p}{2\pi\varepsilon_0\,r^3}$$

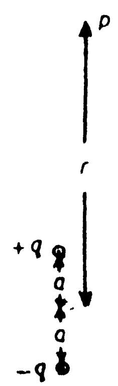


Fig. 27.10

(*b*) Direction of **E** is parallel to **P**

27.11. (*a*) $\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2$

$$E_1=E_2=\frac{q}{4\pi\varepsilon_0\,(a^2+r^2)} \qquad \text{...(1)}$$

The vector sum of $\mathbf{E}_1$ and $\mathbf{E}_2$ points along the perpendicular bisector joining the charges.

$$E=2E_1\cos\theta \qquad \text{...(2)}$$

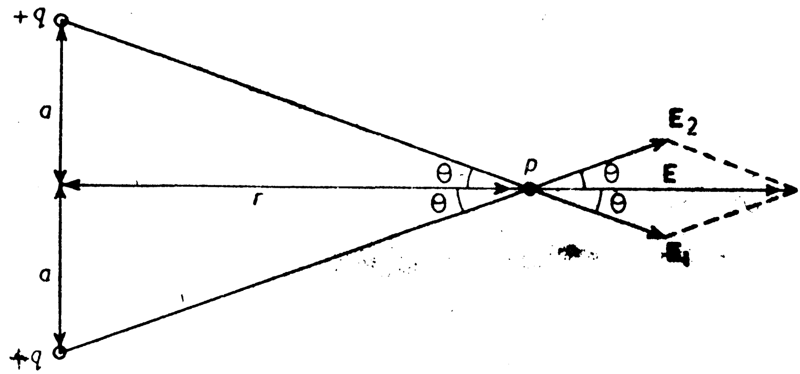


Fig. 27.11

From the geometry of the figure we have

$$\cos\theta = \frac{r}{\sqrt{a^2+r^2}} \qquad \ldots(3)$$

Use (1) and (3) in (2) to find

$$E = \frac{2q}{4\pi\varepsilon_0\,(a^2+r^2)}\;\frac{r}{\sqrt{a^2+r^2}} = \frac{2qr}{2\pi\varepsilon_0(a^2+r^2)^{3/2}}$$

If $r \gg a$, then a^2 can be neglected in comparison with r^2 in the denominator.

$$E = \frac{1}{4\pi\varepsilon_0}\;\frac{2q}{r^2}$$

(*b*) **E** points radially away from the charge axis and lies in the median plane.

(*c*) At great distances the two charges of the same sign behave like a monopole (single charge) for which the electric field is expected to vary as r^{-2}. On the other hand for the dipole with charges q and $-q$ the field is expected to vary as r^{-3}.

27.12. (*a*) Let the point at which the electric field is zero be located at distance x on the charge axis on the right side of $+2q$. The intensity is then given by

$$E = \frac{2q}{x^2} - \frac{5q}{(x+a)^2} = 0$$

$$2(x+a)^2 - 5x^2 = 0$$

whence

$$x = \frac{\sqrt{2}\,a}{\sqrt{5}-\sqrt{2}} = 1.72\,a$$

$$= (1.72)(50\text{ cm}) = 86\text{ cm}.$$

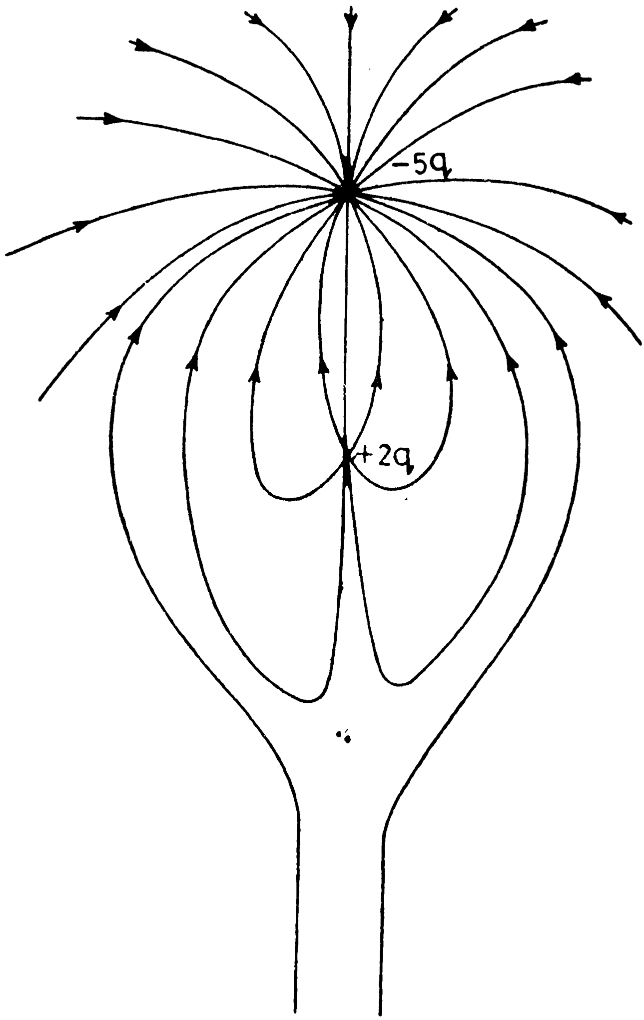


Fig. 27.12

27.13. The electric field at the centre P due to the dipole $+q$, $-q$ is given by

$$E' = 2E_1 \cos\theta \qquad \text{...(1)}$$

where, $$E_1 = \frac{1}{4\pi\varepsilon_0} \frac{q}{\left(\frac{a^2}{4}+\frac{a^2}{4}\right)} = \frac{q}{2\pi\varepsilon_0 a^2} \qquad \text{...(2)}$$

$$\cos\theta = \frac{\frac{1}{2}a}{\sqrt{\frac{a^2}{4}+\frac{a^2}{4}}} = \frac{1}{\sqrt{2}} \qquad \text{...(3)}$$

Use (2) and (3) in (1) to get

$$E' = \frac{2\sqrt{2}\,q}{4\pi\varepsilon_0 a^2} \quad \ldots(4)$$

E' points downward.

The electric field E'' due to the dipole $+2q$, $-2q$ is given by simply replacing q by $2q$ in (4)

$$E' = \frac{2\sqrt{2}\,(2q)}{4\pi\varepsilon_0\, a^2} \quad \ldots(5)$$

It points up.

The resultant field due to the two dipoles is given by

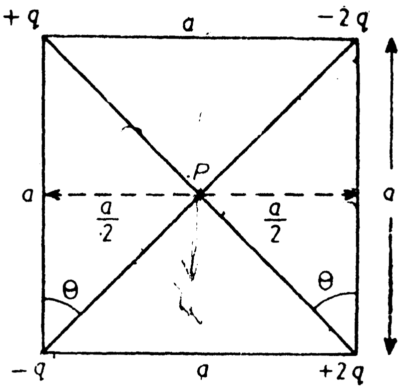


Fig. 27.13

$$\mathbf{E}' = \mathbf{E} + \mathbf{E}'' = \frac{2\sqrt{2}\,(2q)}{4\pi\varepsilon_0\, a^2} - \frac{2\sqrt{2}\,q}{4\pi\varepsilon_0\, a^2} = \frac{2\sqrt{2}\,q}{4\pi\varepsilon_0\, a^2}$$

$$= (2\sqrt{2})\,(9\times10^9 \text{ nt-m}^2/\text{coul}^2)\,(1.0\times10^{-8}\text{ coul})/(0.05\text{m})$$

$$= 1.02\times10^5 \text{ nt/coul}.$$

It points up.

27.14. (*a*) Let the charges q_1 and q_2 be placed at a distance d apart. Let the point P be located at a distance x from q_2 and away from q_1 on the charge axis, where $\mathbf{E}=0$. Assume that q_1 and q_2 are of opposite sign.

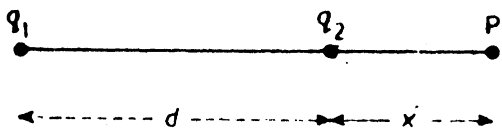


Fig. 27.14

$$E = \frac{q_1}{4\pi\varepsilon_0 (d+x)^2} - \frac{q_2}{4\pi\varepsilon_0 x^2} = 0$$

or

$$q_1 x^2 - q_2\,(d+x)^2 = 0$$

$\therefore$

$$\frac{d+x}{x} = \sqrt{\frac{q_1}{q_2}} \quad \text{or} \quad x = \frac{d\sqrt{q_2}}{\sqrt{q_1} - \sqrt{q_2}} \quad \ldots(1)$$

Thus, charges must be of opposite sign. The nearer charge q_2 must be less in magnitude than the farther charge q_1 since x must be positive. The distance from q_2 is given by (1).

(*b*) As x has to be positive, the second solution for the quadratic equation is unacceptable. Further, at any point away from the charge axis and other than infinity E is always finite. Therefore, no other solution is possible.

27.15. Consider a length dx of the rod at distance x from the center. Then the charge associated with the length dx is $q\,\frac{dx}{l}$. The distance of P from dx is $\sqrt{y^2+x^2}$. At P the electric field due to dx is

$$dE=\frac{q\,dx/l}{4\pi\varepsilon_0(y^2+x^2)}$$

The y-component of the field along the perpendicular bisector of the rod is given by

$$dE_y=dE_y\cos\theta=\frac{q(dx/l)\,y}{4\pi\varepsilon_0(y^2+x^2)^{3/2}}$$

Total perpendicular component of the field is given by

$$E_y=\int dE_\perp=2\int_0^{l/2}\frac{qy\,dx}{4\pi\varepsilon_0(y^2+x^2)^{3/2}}$$

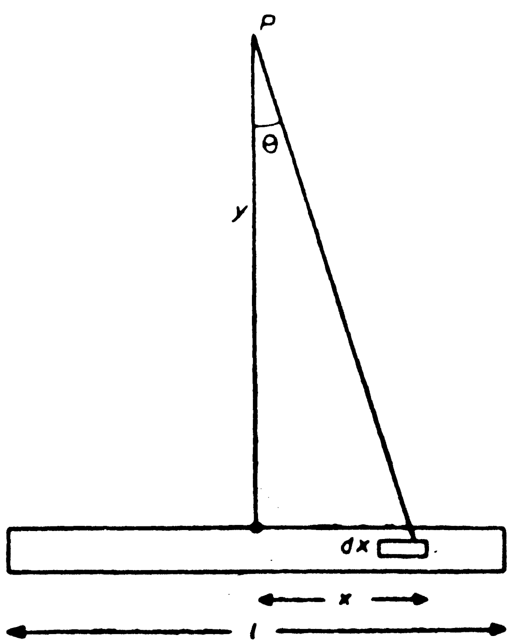


Fig. 27.15

Set
$$x = y \tan\theta$$
$$dx = y \sec^2\theta \, d\theta$$

Then
$$\int_0^{l/2} \frac{dx}{(y^2+x^2)^{3/2}} = \frac{y}{y^3}\int_0^{\theta_0} \frac{\sec^2\theta \, d\theta}{(1+\tan^2\theta)^{3/2}}$$

$$= \frac{1}{y^2}\int_0^{\theta_0} \cos\theta \, d\theta = \frac{\sin\theta_0}{y^2}$$

$$= \frac{l/2}{y2\sqrt{y^2+\dfrac{l^2}{4}}} = \frac{l}{y^2\sqrt{4y^2\times l^2}}$$

Since the x-component of the field would vanish upon integration, we have

$$E = E_y = \frac{2qy}{4\pi\varepsilon_0 l}\,\frac{l}{y^2\sqrt{4y^2+l^2}} = \frac{q}{2\pi\varepsilon_0 y\sqrt{1+(4y^2/l^2)}}$$

Since $\dfrac{q}{l} = \lambda$, the linear charge density (charge per unit length) in he limit $l \to \infty$, we get

$$E = \frac{\lambda}{2\pi\varepsilon_0 y}$$

27.16. Let the coordinate system be located at the center of the circle. Choose the y-axis so that it divides the bent rod into two equal parts. Consider a segment ds of the rod subtended between θ and $\theta+d\theta$. The charge in ds is given by,

$$dq = q\frac{ds}{s}$$

where s is half the length of the rod.

$$dq = q\frac{d\theta}{\frac{1}{2}\theta_0} = 2q\,\frac{d\theta}{\theta_0}$$

The field at O, the center of the circle due to dq is given by

$$dE = \frac{dq}{4\pi\varepsilon_0 a^2} = \frac{2q\,d\theta}{4\pi\varepsilon_0\theta_0 a^2}$$

The y-component of the field due to dq is given by

$$dE_y = dE\cos\theta = \frac{q\cos\theta \, d\theta}{2\pi\varepsilon_0\,\theta_0\,a^2}$$

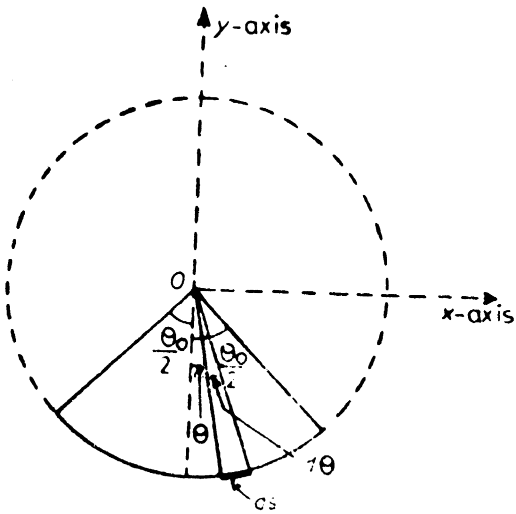


Fig. 27.16

Since the x-component of the field would vanish upon integration, we have

$$E=\int dE_y=\frac{q}{2\pi\varepsilon_0\theta_0\, a^2}\int_0^{\frac{1}{2}\theta_0}\cos\theta\, d\theta=\frac{q}{2\pi\varepsilon_0\theta_0\, a^2}\sin\theta\Big|_0^{\frac{1}{2}\theta_0}$$

$$=\frac{q\sin\left(\frac{1}{2}\theta_0\right)}{2\pi\varepsilon_0\theta_0\, a^2}$$

Consider an element of area ds in the form of a circular strip symmetrically placed over the inner surface of the hemisphere. The radius of the strip is $a \sin\theta$ and its width is $(a\, d\theta)$ where θ is the polar angle.

$$ds=2\pi(a\sin\theta)\,(a d\theta)=2\pi a^2\sin\theta\, d\theta$$

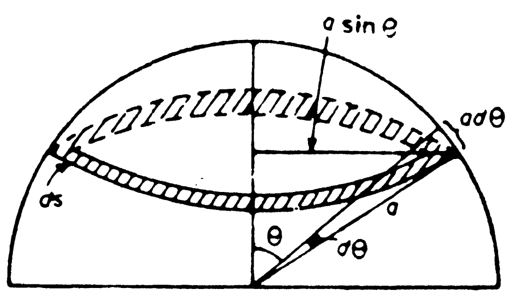


Fig. 27.17

As the area of the hemisphere is, $A=2\pi a^2$, the charge spread over this strip is

$$dq=q\frac{ds}{A}=\frac{q(2\pi a^2 \sin\theta\, d\theta)}{2\pi a^2}=q\sin\theta\, d\theta$$

The electric field acting at O, the center of the hemisphere is

$$dE=\frac{dq}{4\pi\varepsilon_0 a^2}=\frac{q\sin\theta d\theta}{4\pi\varepsilon_0\, a^2}$$

The y-component of the field is

$$dE_y=dE\cos\theta$$

As the x-component of the field vanishes upon integration the total field E is given by

$$E=\int dE_y=\frac{q}{4\pi\varepsilon_0\, a^2}\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta$$

$$=\frac{q}{4\pi\varepsilon_0\, a^2}\int_0^1 \sin\theta\, d(\sin\theta)=\frac{q}{8\pi\varepsilon_0\, a^2}$$

It points along the axis of symmetry and away from the hemispere.

27.18. The combined electric field due to the charge $+q$, $-2q$, $+q$ at distance $(r-a)$, r and $(r+a)$ respectively, is given by

$$E=\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{(r-a)^2}-\frac{2q}{r^2}+\frac{q}{(r+a)^2}\right]$$

$$=\frac{2qa^2\,(3r^2-a^2)}{4\pi\varepsilon_0\, r^2(r^2-a^2)^2}$$

Since $r \gg a$, $3r^2-a^2 \simeq 3r^2$ and $r^2-a^2\simeq r^2$.

Also, $\quad Q=2qa^2$

$\therefore \quad E=\dfrac{3Q}{4\pi\varepsilon_0 r^4}$

27.19. The electric field on the axis of the charged ring is given by

$$E=\frac{1}{4\pi\varepsilon_0}\frac{qx}{(a^2+x^2)^{3/2}}$$

where a is the radius of the ring and x is the distance of the electron from the center of the ring along the axis.

For $x \ll a$, we can neglect x^2 in comparison with a^2 in the denominator.

$$E=\frac{qx}{4\pi\varepsilon_0\, a^3}$$

The force F acting on the electron is

$$F=-eE=-\frac{eqx}{4\pi\varepsilon_0 a^3}$$

$\therefore$ Acceleration, $$a=\frac{F}{m}=-\frac{eqx}{4\pi\varepsilon_0 ma^3}=-\omega^2 x$$

As the acceleration is directly proportional to the displacement but oppositely directed, the motion is simple harmonic, with angular frequency

$$\omega=\sqrt{\frac{eq}{4\pi\varepsilon_0 ma^3}}$$

27.20. The electric field is given by

$$E=\frac{qx}{4\pi\varepsilon_0(a^2+x^2)^{3/2}} \qquad \cdots(1)$$

Differentiating (1) with respect to x and setting $\frac{dE}{dx}=0$, to find the maximum value of E,

$$\frac{dE}{dx}=\frac{d\left[(a^2+x^2)^{3/2}-x.\,2x\,\frac{3}{2}(a^2+x^2)^{\frac{1}{2}}\right]}{4\pi\varepsilon_0(a^2+x^2)^3}=0$$

$$(a^2+x^2)^{3/2}\;(a^2+x^2)^{\frac{1}{2}}-3x^2=0$$

Factoring

$$(a^2+x^2)^{1/2}\;(a^2+x^2-3x^2)^{\frac{1}{2}}=0$$

Since $$(a^2+x^2)^{\frac{1}{2}}\neq 0,$$

we have $$a^2-2x^2=0$$

or $$x=\frac{a}{\sqrt{2}}.$$

Thus, the maximum value of E occurs at $x=a/\sqrt{2}$.

27.21. Consider an element of the ring of length ds located at the top of the ring shown in the textbook Fig. 27.10. The element of charge associated with it is

$$dq_1=q_1\frac{ds}{\pi a}$$

where q_1 is the charge in the upper half of the circumference. The differential electric field at P is given by

$$dE_1=\frac{1}{4\pi\varepsilon_0}\frac{dq_1}{r^2}=\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 ds}{\pi a}\right)\frac{1}{a^2+x^2}$$

where x is the distance of P from the center of the ring whilst r is the radial distance of P from the circumference of the ring. Note that for a fixed point P, x has the same value for all charge elements and is not a variable.

(*a*) The *x*-component (along the axis) of the field is given by

$$E_1(x)=\int dE_1\cos\theta$$

$$=\int\frac{1}{4\pi\varepsilon_0}\quad\frac{q_1\,ds}{\pi a\,(a^2+x^2)}\quad\frac{x}{\sqrt{a^2+x^2}}$$

$$=\frac{q_1x}{4\pi\varepsilon_0\,(\pi a)\,(a^2+x^2)^{3/2}}\int ds$$

But $\int ds=\pi a$, half of the circumference of the ring, so that

$$E_1(x)=\frac{q_1x}{4\pi\varepsilon_0\,(a^2+x^2)^{3/2}}$$

Similarly, due to charge q_2 in the lower half of the ring,

$$E_2(x)=\frac{q_2x}{4\pi\varepsilon_0\,(a^2+x^2)^{3/2}}$$

Field along the axis due to the entire ring containing charge $q=q_1+q_2$ is

$$E(x)=E_1(x)+E_2(x)=\frac{x\,(q_1+q_2)}{4\pi\varepsilon_0\,(a^2+x^2)^{3/2}}=\frac{qx}{4\pi\varepsilon_0\,(a^2+x^2)^{3/2}}$$

since $E_1(x)$ and $E_2(x)$ point in the same direction.

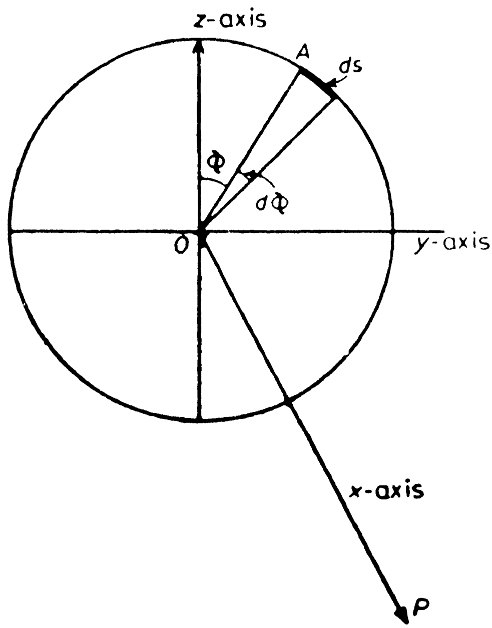


Fig 27.M

(*b*) The transverse component (perpendicular to the axis) of the field due to q_1 is given by

$$E_1(T) = \int dE_1 \sin\theta$$

$$= \int \frac{1}{4\pi\varepsilon_o} \frac{q_1\, ds}{\pi a\,(a^2+x^2)} \frac{a}{\sqrt{a^2+x^2}}$$

$$= \frac{q_1 a}{4\pi\varepsilon_o\,(\pi a)\,(a^2+x^2)^{3/2}} \int ds$$

Since $\int ds = \pi a$

$$E_1(T) = \frac{q_1 a}{4\pi\varepsilon_o\,(a^2+x_2)^{3/2}}$$

Let ds be defined by the azimuthal angles ϕ and $\phi+d\phi$. The angle ϕ being measured with respect to the z-axis. The transverse component will lie in the direction OA. Hence, its projection on the z-axis averaged over the azimuth angle gives the z-component.

$$E_1(z) = E_1(T) \int_0^{\pi/2} \cos\phi\, \frac{d\phi}{\pi/2} = \frac{2}{\pi} E_1(T) \sin\phi \Big|_0^{\pi/2}$$

$$= \frac{2}{\pi} E_1(T) = \frac{q_1\, a}{2\pi^2\varepsilon_o\,(a^2+x^2)^{3/2}}$$

For the other half of the circumference, q_2 will contribute to the field in the opposite direction.

$$E_2(z) = -\frac{q_2 a}{2\pi^2\varepsilon_o\,(a^2+x^2)^{3/2}}$$

Net component of electric field perpendicular to the axis in a fixed direction is

$$E(z) = E_1(z) + E_2(z) = \frac{(q_1-q_2)a}{2\pi^2\varepsilon_o\,(a^2+x^2)^{3/2}}$$

27.22. Consider a ring of radius x and width dx, concentric with the disk. The charge in the ring is

$$dq = (2\pi x dx)\sigma$$

The electric field at P at distance r along the axis from the center of the disk, due to element of charge dq is given by

$$dE = \frac{dq}{4\pi\varepsilon_o R^2} = \frac{(2\pi x\, dx)\sigma}{4\pi\varepsilon_o R^2} = \frac{\sigma x\, dx}{2\varepsilon_o\,(r^2+x^2)}$$

The component of field along the axis due to dq is

$$dEr = dE\cos\theta = \frac{\sigma x\, dx}{2\varepsilon_o\,(r^2+x^2)} \frac{r}{\sqrt{r^2+x^2}} = \frac{\sigma r x dx}{2\varepsilon_o\,(r^2+x^2)^{3/2}}$$

The component of the field in the direction perpendicular to the axis vanishes upon integration.

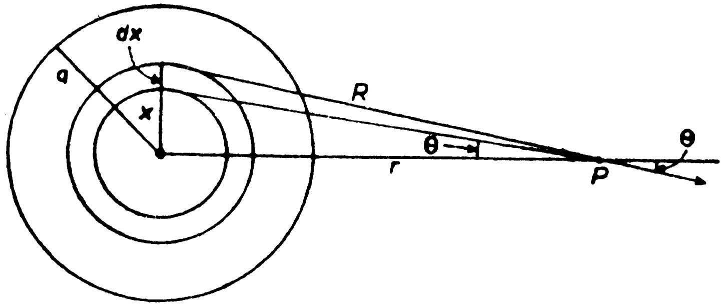


Fig. 27.22

The electric field on the axis of the disk at distance r is given by

$$E=\int dEr=\frac{\sigma r}{2\varepsilon_o}\int_0^a \frac{x\,dx}{(r^2+x^2)^{3/2}}$$

Set $\qquad x=r\tan\theta$

$$dx=r\sec^2\theta\,d\theta$$

The integral $\qquad I=\int\frac{(r\tan\theta)(r\sec^2\theta\,d\theta)}{(r^2+r^2\tan^2\theta)^{3/2}}=\frac{1}{r}\int\sin\theta\,d\theta$

$$E=-\frac{1}{r}\cos\theta\Big|_0^{r/\sqrt{a^2+r^2}}=\frac{1}{r}\left[1-\frac{r}{\sqrt{a^2+r^2}}\right]$$

Therefore, $\qquad E=\frac{\sigma}{2\varepsilon_o}\left[1-\frac{r}{\sqrt{a^2+r^2}}\right]$

27.23. The field due to charge $+q$ at P is

$$E_1=\frac{q}{4\pi\varepsilon_o r_1^2}$$

The x-component of the field is $E_1(x)=E_1\cos\theta_1=E_1(x/r_1)$

$$=\frac{qx}{4\pi\varepsilon_o r_1^3}=\frac{qx}{4\pi\varepsilon_o[x^2+(y-a)^2]^{3/2}}$$

Similarly, due to $-q$,

$$E_2(x)=-\frac{qx}{4\pi\varepsilon_o[x^2+(y+a)^2]^{3/2}}$$

Neglecting the term a^2 in the denominators,

$$E(x)=E_1(x)+E_2(x)$$

$$=\frac{qx}{4\pi\varepsilon_o}\left\{\frac{1}{(x^2+y_2-2ya)^{3/2}}-\frac{1}{(x^2+y^2+2ya)^{3/2}}\right\}$$

$$=\frac{qx}{4\pi\varepsilon_o(x^2+y^2)^{3/2}}\left[\left(1-\frac{2ya}{x^2+y^2}\right)^{-\frac{3}{2}}-\left(1+\frac{2ya}{x^2+y^2}\right)^{-\frac{3}{2}}\right]$$

$$=\frac{6q\ xya}{4\pi\varepsilon_0\ (x^2+y^2)^{5/2}}$$

where we have retained terms linear in a and ignored higher order terms. Setting $p=2aq$

$$E_x=\frac{1}{4\pi\varepsilon_0}\ \frac{3pxy}{(x^2+y^2)^{5/2}}$$

The y-component of field due to charge $+q$ is

$$E_1(y)=E_1\sin\theta_1=E_1\frac{(y-a)}{r_1}=\frac{q}{4\pi\varepsilon_0}\ \frac{(y-a)}{[x^2+(y-a)^2]^{3/2}}$$

Similarly, due to charge $-q$, the y-component is

$$E_2(y)=-\frac{q(y+a)}{4\pi\varepsilon_0[x^2+(y+a)^2]^{3/2}}$$

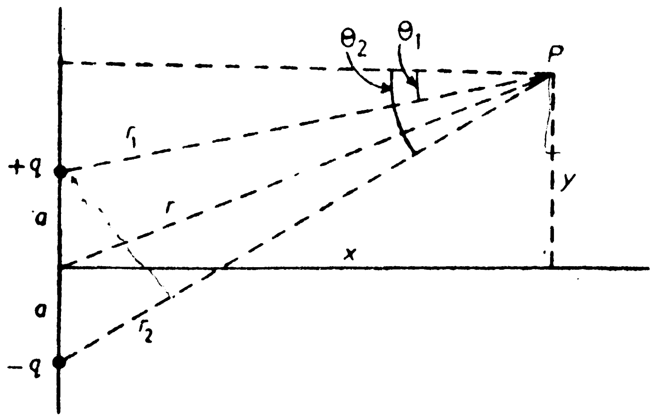


Fig. 27.23

Neglecting a^2 in the denominators,

$$E(y)=E_1(y)+E_2(y)$$

$$=\frac{q\left[(y-a)\left(1-\frac{2ya}{x^2+y^2}\right)^{-\frac{3}{2}}-(y+a)\left(1+\frac{2ya}{x^2+y^2}\right)^{-\frac{3}{2}}\right]}{4\pi\varepsilon_0(x^2+y^2)^{3/2}}$$

$$=\frac{q\left[(y-a)\left(1+\frac{3ya}{x^2+y^2}\right)-(y+a)\left(1-\frac{3ya}{x^2+y^2}\right)\right]}{4\pi\varepsilon_0(x^2+y^2)^{3/2}}$$

$$=\frac{q(2a)(2y^2-x^2)}{4\pi\varepsilon_0(x^2+y^2)^{5/2}}$$

Where we have neglected terms involving a^2 and higher order terms Setting $p=2aq$

$$E(y)=\frac{p(2y^2-x^2)}{4\pi\varepsilon_0(x^2+y^2)^{5/2}}$$

27.24. Weight of electron, $F=mg=(9.1\times10^{-31}\text{g})\ (9.8 \text{ meter/sec}^2)$
$=8.92\times10^{-30}\text{nt}$

Electron charge, $e=1.6\times10^{-19}$ coul

Electron field $E=\dfrac{F}{e}=\dfrac{8.92\times10^{-30}\text{ nt}}{1.6\times10^{-19}\text{ coul}}=5.6\times10^{-11}$ nt/coul

in the downward direction

(*b*) weight of α-particle $F=\text{mg}=(6.68\times10^{-27}\text{ kg})\ (9.8 \text{ meter/sec}^2)$
$=6.55\times10^{-26}$ nt

Charge of α-particle, $q=2\ e=3.2\times10^{-19}$ coul

Electric field, $E=\dfrac{F}{q}=\dfrac{6.55\times10^{-26}\text{ nt}}{3.2\times10^{-19}}=2\times10^{-7}$ nt/coul

in the, upward direction

27.25. (*a*) $E=\dfrac{F}{q}=\dfrac{3\times10^{-6}\text{ nt}}{2\times10^{-9}\text{ coul}}=1.5\times10^3$ nt/coul

(*b*) $F=Ee=(1.5\times10^3\text{ nt/coul})\ (1.6\times10^{-19}\text{ coul})$
$=2.4\times10^{-16}$ nt (up)

(*c*) Gravitational force,
$F=mg=(1.67\times10^{-27}\text{ kg})\ (9.8 \text{ meter/sec}^2)$
$=1.6\times10^{-26}$ nt

(*d*) Ratio of the electric to the gravitational force,
$\dfrac{Ee}{mg}=\dfrac{2.4\times10^{-16}\text{ nt}}{1.6\times10^{-26}\text{ nt}}=1.5\times10^{10}$

27.26. (*a*) Acceleration, $a=\dfrac{F}{m}=\dfrac{Ee}{m}$
$=\dfrac{(10^6\text{ nt/coul})\ (1.6\times10^{-19}\text{ coul})}{9.1\times10^{-31}\text{ kg})}=1.8\times10^{17}$ meter/sec

(*b*) Initial velocity $v_o=0$

Final velocity $v=0.1c=\left(\dfrac{1}{10}\right)\left(3\times10^8\text{ meter/sec}\right)=3\times10^7$ m/sec

$t=\dfrac{v-v_o}{a}=\dfrac{3\times10^7\text{ meter/sec}}{1.8\times10^{17}\text{ meter/sec}^2}=1.7\times10^{-10}$ sec

(*c*) High speeds attained by the particles owing to intense electric fields limit the applicability of Newtonian mechanics.

27.27. (*a*) Force, $F=Ee=(1.0\times10^{-3}\text{ nt/coul})\ (1.6\times10^{-19}\text{ coul})$
$=1.6\times10^{-16}$ nt

Acceleration, $a=\dfrac{F}{m}=\dfrac{1.6\times10^{-16}\text{ nt}}{9.1\times10^{-31}\text{ kg}}=1.76\times10^{14}$ meter/sec^2

$v^2=v_o^2-2as$

$s=\dfrac{v_o^2-v^2}{2\ a}=\dfrac{(5\times10^6\text{ meter/sec})^2-0}{(2)(1.76)\times10^{14}\text{ meter/sec}^2)}$
$=7.1\times10^{-2}$ meter$=7.1$ cm.

(*b*) $v=v_o-at$

$$t=\frac{v_o-v}{a}=\frac{(5\times10^6\text{ meter/sec})-0}{1.76\times10^{14}\text{ meter/sec}^2}=2.8\times10^{-8}\text{ sec.}$$

(*c*) $v^2=v_o^2-2\,as$

$$=(5\times10^6\text{ meter/sec})^2-2\,(1.76\times10^{14}\text{ meter/sec}^2)$$
$$(0.8\times10^{-2}\text{ meter})=22.18\times10^{12}\text{ (meter/sec)}^2$$

$\therefore$ $v=4.7\times10^6$ meter/sec

Fraction of kinetic energy lost

$$\frac{\Delta K}{K_o}=\frac{K_o-K}{K_o}=1-\frac{K}{K_o}=1-\frac{v^2}{v_o^2}$$
$$=1-[(4.7\times10^6\text{ meter/sec})/(5.0\times10^6\text{ meter/sec})]^2$$
$$=0.116\text{ or }11.6\%.$$

27.28. (*a*) Acceleration, $a=\dfrac{F}{m}=\dfrac{Ee}{m}$

$$=\frac{(2.0\times10^3\text{ nt/coul})(1.6\times10^{-19}\text{ coul})}{(9.1\times10^{-31}\text{ kg})}$$
$$=3.5\times10^{14}\text{ meter/sec}^2$$
$$=3.5\times10^{16}\text{ cm/sec}^2$$

Equation of the trajectory is

$$y=(\tan\theta_o)\,x-\frac{ax^2}{2(v_o\cos\theta_o)^2}$$

$$2\text{ cm}=(\tan 45^\circ)\,x-\frac{(3.5\times10^{16}\text{ cm/sec}^2)\,x^2}{2\,(6.0\times10^8\text{ cm/sec})^2\,(\cos 45^\circ)^2}$$

Simplifying,

$$3.5x^2-36x+72=0 \qquad \ldots(1)$$

As the discriminant (b^2-4ac) of the above quadratic equation is positive, the roots will be real. Hence, the electron will strike the upper plate.

(*b*) Eq. (1) yields the roots $x=2.7$ cm and 7.6 cm. The electron will strike the upper plate at a distance 2.7 cm from the left edge Here the second solution ($x=7.6$ cm) is not realized.

27.29. The field at *A* is

$$E=\frac{q}{4\pi\varepsilon_o}\left[\frac{1}{z^2}-\frac{1}{(l-z)^2}\right]$$

where l is the distance between the two charges (Fig. 27.29 Differentiating E with respect to z

$$\frac{dE}{dz}=-\frac{2q}{4\pi\varepsilon_o}\left[\frac{1}{z^3}+\frac{1}{(l-z)^3}\right]$$

Set $z=\dfrac{l}{2}$

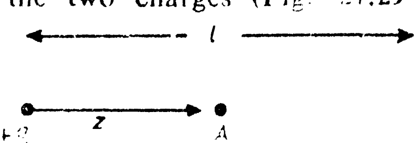


Fig. 27.29

Then, $$\frac{dE}{dz} = -\frac{8q}{\pi\varepsilon_0 l^3}$$

Now, the magnitude of the force on an electric dipole moment placed in a non-uniform electric field E is given by the relation

$F = p\frac{\partial E}{\partial z}$ where p is the dipole moment. But, $\frac{\partial E}{\partial z}$ at $z = \frac{a}{2}$ is non-zero. Hence, force would be exerted on a dipole placed at $z = \frac{a}{2}$ notwithstanding the fact that $E=0$ at this point.

27.30. (*a*) Balancing the electric force by the weight of the oil drop,

$$Eq = mg$$

$$m = \frac{4}{3}\pi R^3\rho = \frac{4}{3}\pi\left(1.64\times10^{-6}\text{ meter})^3 (851\text{ kg/meter}^3\right)$$

$$= 1.57\times10^{-14}\text{ kg}$$

$$q = \frac{mg}{E} = \frac{(1.57\times10^{-14}\text{ kg})(9.8\text{ meter/sec}^2)}{1.92\times10^5\text{ nt/coul}}$$

$$= 8.0\times10^{-19}\text{ coul.}$$

But $\quad q = ne$

where e is the charge of the electron and n is the number of electrons. Therefore,

$$n = \frac{q}{e} = \frac{8\times10^{-19}\text{ coul}}{1.6\times10^{-19}\text{ coul}} = 5$$

(*b*) First, electrons cannot be seen. Secondly, in order to balance, the fields to be employed would be too small.

27.31. Taking the differences between various measured charges arranged in the ascending order, we have in units of 10^{-19} coul,

1.6374 ; 3.296 ; 1.63 ; 3.35
1.600 ; 1.63 ; 3.18 ; 3.24

The above figures are seen to be in multiples of the elementary charge of about 1.6×10^{-19} coul. We therefore find the mean elementary charge from

$$\langle e\rangle = \tfrac{1}{8}[1.637 + \tfrac{1}{2}(3.296) + 1.630 + \tfrac{1}{2}(3.350) + 1.600$$
$$+ 1.630 + \tfrac{1}{2}(3.180) + \tfrac{1}{2}(3.240)]\times10^{-19}\text{ coul}$$
$$= 1.63\times10^{-19}\text{ coul.}$$

27.32. (*a*) Gravitational force on the sphere must be balanced by electric force.

$$qE = mg$$

$$q = \frac{mg}{E} = \frac{(0.453\text{ kg})(9.8\text{ meter/sec}^2)}{150\text{ nt/coul}}$$
$$= 0.03\text{ coul}$$

The charge must be positive.

(*b*) $$r_o=\left(\frac{3\,M}{4\pi\rho}\right)^{\frac{1}{3}}=\left[\frac{(3)\,(0.453\text{ kg})}{(4\pi)(2000\text{ kg/meter}^3)}\right]^{1/3}$$

$$=0.0378\text{ meter}$$

The electric field on the surface of the sphere is

$$E=\frac{Q}{4\pi\varepsilon_o r_o^{\,2}}=\frac{(9\times10^9\text{ nt-m}^2/\text{coul}^2)\,(0.03\text{ coul})}{(0.0378\text{ meter})^2}$$

$$=1.9\times10^{11}\text{ volt/m}$$

a value which is much in excess of 3×10^6 volt/meter for the electrical breakdown in air. The sphere itself may get blown off owing to intense electric field.

SUPPLEMENTARY PROBLEMS

S.27.1. The electric force acting on the sphere of charge $+q$ is

$$F_e=qE$$

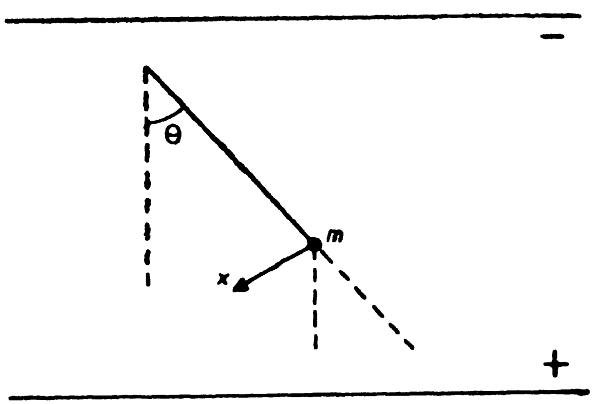


Fig. S.27.1

Let the sphere be displaced through a small angle θ from the equilibrium position, the linear displacement being x along the arc. While the sphere is attracted down due to gravitational force F_g, it is repelled by the positive charge on the lower plate. The net component of force along x is

$$F=-(F_g+F_e)\sin\theta=-(mg-qE)\sin\theta$$

The negative sign has been introduced as the restoring force acts in the direction opposite to the displacement. As θ is small

$$\sin\theta=\theta=\frac{x}{l}$$

$$\therefore\qquad F=ma=-\left(mg-qE\right)\frac{x}{l}$$

Acceleration is given by

$$a=\frac{F}{m}=-\left(g-\frac{qE}{m}\right)\frac{x}{l}=-\omega^2 x \qquad \ldots(1)$$

with $\omega^2=\frac{1}{l}\left(g-\frac{qE}{m}\right)$

Equation (1) is that of simple harmonic motion as the acceleration is proportional to displacement and is oppositely directed.

$$\therefore \qquad T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{l}{g-(qEm)}} \qquad \ldots(2)$$

If the lower plate is negatively charged the net force due to gravitation and electric field would be enhanced as the forces act in the same direction, the time period would then be

$$T=2\pi\sqrt{\frac{l}{g+(qE/m)}} \qquad \ldots(3)$$

Note that by setting either $q=0$ or $E=0$ in (2) or (3), we would get the familiar formula for simple pendulum viz. $T=2\pi\sqrt{l/g}$.

S.27.2. (*a*) Owing to symmetry the magnitude of forces F^+ and F^- acting on the charge q will be equal (Fig. S.27.2). The resultant will be obtained by completing the parallelogram and drawing the diagonal. From the geometry of the figure, it is obvious that the resultant F points in the direction antiparallel to that of the dipole moment. (Recall that the dipolement is directed from negative charge towards positive charge).

(*b*) The direction of force on the dipole will be opposite to that of F i.e. parallel to the dipole moment.

(*c*) Magnitude of force on the dipole is

$$F=5\times 10^{-6} \text{ nt}.$$

(*b*) and (*c*) follow from Newton's third law of motion viz., action and reaction are equal and opposite.

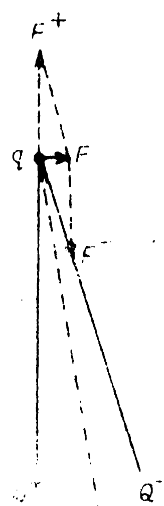


Fig. S.27.2

S.27.3. Consider the differential element of charge dq in the element of arc ds for the upper half of the semi-circle. The electric field at P due to this elementary charge points along the the radius vector and is indicated by dE^+ in Fig. S.27.3. Resolve dE^+ into dE_x^+ and dE_y^+ along the x and y-axes respectively

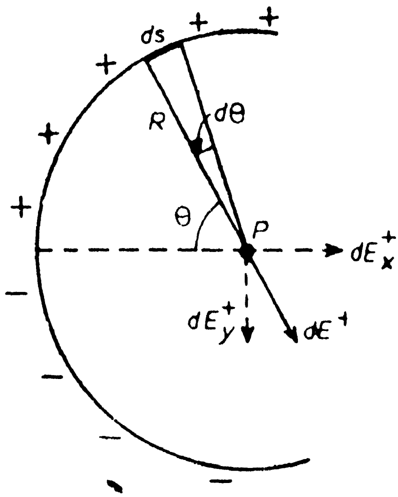


Fig. S.27.3

$$ds = R\,d\theta \qquad \ldots(1)$$

$$dq = \frac{Q\,ds}{\frac{1}{2}\pi R} = \frac{2Q\,d\theta}{\pi} \qquad \ldots(2)$$

where use has been made of (1)

$$dE^+ = \frac{dq}{4\pi\varepsilon_0 R^2} \qquad \ldots(3)$$

$$dE_y^+ = dE^+ \sin\theta = \frac{dq \sin\theta}{4\pi\varepsilon_0 R^2} = \frac{Q \sin\theta\,d\theta}{2\pi^2\varepsilon_0 R^2} \qquad \ldots(4)$$

where use has been made of (3) and (2).

Integrating (4),

$$E_y^+ = \frac{Q}{2\pi^2\varepsilon_0 R^2} \int_0^{\pi/2} \sin\theta d\theta = \frac{Q}{2\pi^2\varepsilon_0 R^2}$$

in the downward direction.

It is found that due to charge $-Q$ spread over the lower half of the semi-circle,

$$E_y^- = \frac{Q}{2\pi^2\varepsilon_0 R^2}$$

again in the downward direction.

Electric field due to both $+Q$ and $-Q$,

$$E_y = E_y^+ + E_y^- = \frac{(2)\,Q}{2\pi^2\varepsilon_0 R^2} = \frac{Q}{\pi^2\varepsilon_0 R^2}$$

However, the x-components E_x^+ and E_x^- due to the two charges cancel each other. Hence electric field at P is

$$E = E_y = \frac{Q}{\pi^2\varepsilon_0 R^2}$$

S.27.4. Consider a differential element of length dx at distance x from A. The element of charge associated with dx is

$$dq = \lambda\,dx \qquad \ldots(1)$$

The magnitude of the field contribution due to charge element dq at P is

$$dE=\frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}=\frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{(R^2+x^2)} \qquad ..(2)$$

where use has been made of (1).

The vector $d\mathbf{E}$ has the components

$$dE_x=-dE\sin\theta \qquad ...(3)$$

$$dE_y=-dE\cos\theta \qquad ...(4)$$

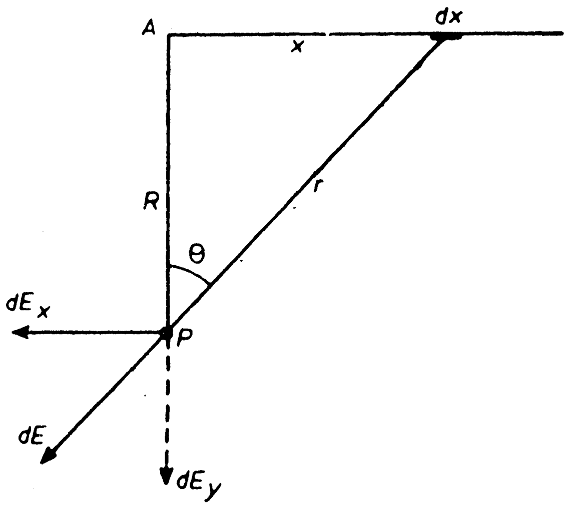


Fig. S.27 4]

The minus signs indicate that dE_x and dE_y point respectively in the negative x and y-directions.

The resultant x-component of the field is obtained by integrating (3).

$$E_x=\int dE_x=-\int dE\sin\theta$$

$$E_x=-\frac{\lambda}{4\pi\varepsilon_0}\int_0^\infty\frac{x\,dx}{(R^2+x^2)^{3/2}} \qquad ...(5)$$

where use has been made of (2) and the relation

$$\sin\theta=x/\sqrt{R^2+x^2}.$$

With the change of variable $x=R\tan\theta$, and $dx=R\sec^2\theta\,d\theta$, we find

$$E_x=\int dE_x=-\frac{\lambda}{4\pi\varepsilon_0 R}\int_0^{\pi/2}\sin\theta d\theta=-\frac{\lambda}{4\pi\varepsilon_0 R} \qquad ...(6)$$

Similarly, we have for the y-component,

$$dE_y = -dE \cos\theta = -\frac{\lambda R}{4\pi\varepsilon_0} \frac{dx}{(R^2+x^2)^{3/2}} \quad ...(7)$$

With the change of variable $x = R\tan\theta$ and $dx = R\sec^2\theta\, d\theta$, we find

$$E_y = dE_y = -\frac{\lambda}{4\pi\varepsilon_0 R}\int_0^{\pi/2}\cos\theta\, d\theta = -\frac{\lambda}{4\pi\varepsilon_0 R} \quad ...(8)$$

The angle which the vector **E** makes with the y-direction is given by

$$\tan\theta_o = \frac{E_x}{E_y} = 1 \quad ...(9)$$

where use has been made of (6) and (8). From (9) we find $\theta_o = 45°$.

S.27.5. The torque acting on the dipole is given by

$$\tau = \mathbf{P}\times\mathbf{E} = pE\sin\theta = pE\theta \quad ...(1)$$

where small angles have been considered so that $\sin\theta = \theta$.

Also, $$\tau = I\alpha = I\frac{d^2\theta}{dt^2} \quad ...(2)$$

There is a restoring torque acting on the dipole which enables it to return to the equilibrium position. Comparing (1) and (2)

$$I\frac{d^2\theta}{dt^2} = -pE\theta \quad ...(3)$$

Where we have inserted the minus sign in the right side since the restoring torque acts in the direction opposite to the angular displacement.

$$\frac{d^2\theta}{dt^2} = -\frac{pE\theta}{I} = -\frac{K\theta}{I} \quad ...(4)$$

Where k is the torsional constant with, $k = pE$. The period of oscillation for the torsional pendulum is given by

$$T = 2\pi\sqrt{\frac{I}{k}}$$

and the frequency of oscillations is given by

$$\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{I}} = \frac{1}{2\pi}\sqrt{\frac{pE}{I}}$$

S.27.6. Two forces $+F_1$ and $-F_2$ act in the oppposite directions as shown in Fig. S.27,6.

$$F=F_1-F_2$$

Net force

$$F_2=qE$$

where E is the field at the negative charge. As the field is varying in the vertical direction, the field at the positive charge will be

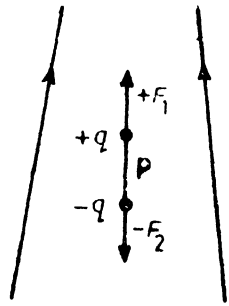


Fig. S.27.6

$$E+dE=E+\frac{\partial E}{\partial y}dy$$

$\therefore \quad F_1=q\,(E+dE)$

$$=q\left(E+\frac{\partial E}{\partial y}dy\right)$$

Set $\quad dy=2a$, the charge separation distance.

Then $\quad F_1=qE+(2\,aq)\,\dfrac{\partial E}{\partial y}$

$$=qE+p\frac{\partial E}{\partial y}$$

$\therefore \quad F=qE+p\,\dfrac{\partial E}{\partial y}-qE=p\,\dfrac{\partial E}{\partial y}$

pointing upward.

S.27.7. The field E due to dipole at a point a distance r, along the perpendicular bisector of the line joining the charges is given by

$$E=\frac{2\,aq}{4\,\pi\varepsilon_0\,r^3} \qquad \ldots(1)$$

where $2a$ is the distance between equal and opposite charges q of the dipole and $r \gg a$. The magnitude of field at P due to the dipole closer to P is

$$E_1=\frac{2\,aq}{4\pi\varepsilon_0\,(R-a)^3} \qquad \ldots(2)$$

pointing down; that due to the dipole which is farther is

$$E_2=\frac{2\,aq}{4\,\pi\varepsilon_0\,(R+a)^3} \qquad \ldots(3)$$

pointing up.

Therefore, the net field is

$$E_{(net)}=E_1-E_2=\frac{2\,aq}{4\,\pi\varepsilon_0}\left[\frac{1}{(R-a)^3}-\frac{1}{(R+a)^3}\right]$$

$$=\frac{2\,aq}{4\,\pi\varepsilon_0}\,\frac{2a\,(3R^2+a^2)}{(R^2-a^2)^3}$$

Neglect a^2 in comparison with R^2. Then

$$E_{(net)}=\frac{3qa^2}{\pi\varepsilon_0\,R^4}$$

28 GAUSS'S LAW

28.1 The flux is given by

$$\phi_E = \oint \mathbf{E} . d\mathbf{S}$$

The field **E** makes an angle θ with the element of area dS which is equal to $2\pi R^2 \sin\theta \, d\theta$ (Fig 28.1).

$$\phi_E = \oint (E)\,(2\pi R^2 \sin\theta \, d\theta) \cos\theta$$

$$= \pi R^2 E \int_0^{\pi/2} (2 \sin\theta \cos\theta) \, d\theta$$

$$= \pi R^2 E \int_0^{\pi/2} \sin 2\theta \, d\theta$$

$$= -\frac{\pi R^2 E}{2} \cos 2\theta \Big|_0^{\pi/2}$$

$$= \pi R^2 E$$

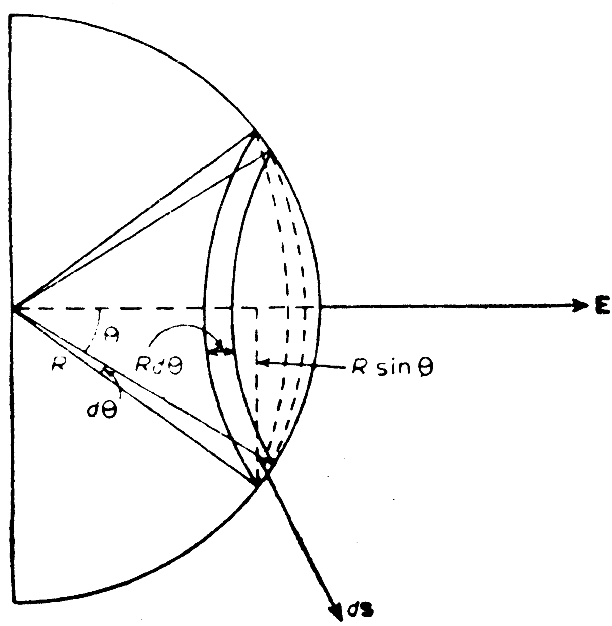


Fig 28.1

28.2. The flux ϕ_E can be written as the sum of three terms, an integral over (*a*) the lower cap, (*b*) the cylindrical surface and (*c*) the upper cap (Fig. 28. 2).

$$\phi_E = \oint \mathbf{E}.d\mathbf{S}$$

$$= \int_{(a)} \mathbf{E}.d\mathbf{S} + \int_{(b)} \mathbf{E}.d\mathbf{S} + \int_{(c)} \mathbf{E}.d\mathbf{S}$$

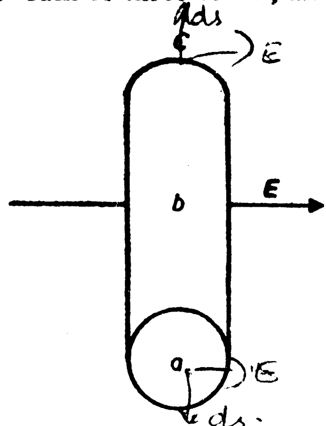


Fig. 28.2

Now, for the caps,

$$\int_{(a)} \mathbf{E}.d\mathbf{S} = \int_{(c)} \mathbf{E}.d\mathbf{S} = 0$$

because $\theta = 90°$, $\mathbf{E}.d\mathbf{S} = 0$ for all points on the caps.

For part (*b*), the curved surface may be divided about a plane perpendicular to **E** and dividing the cylinder into two halves, right and left. For right half $\theta < 90°$ whilst for the left half $90 < \theta < 180°$ so that for reasons of symmetry the contribution to the intergral for the entire curved surface vanishes. Consequently $\phi_E = 0$.

28.3. $\phi_E = \oint \mathbf{E}.d\mathbf{S}$

$$= \oint E \cos \theta dS$$

$$= E \cos \theta \int dS$$

$$= EA \cos \theta$$

where we have taken $\cos \theta$ outside the integral. as θ is constant.

28.4. Flux ϕ_E and the net charge q enclosed by the Gaussian surface are given by the relation

$$\varepsilon_o \phi_E = q$$

$$\therefore \quad \phi_E = \frac{q}{\varepsilon_o} = \frac{1.0 \times 10^{-6} \text{ coul}}{8.9 \times 10^{-12} \text{ coul}^2/\text{nt-m}^2} = 1.1 \times 10^5 \text{ nt-m}^2/\text{coul}$$

28.5. As the net charge enclosed by S_1, S_3, S_5 is positive, ϕ_E is also positive.

As the net charge enclosed by S_2 is negative, ϕ_E is negative.

As the net charge enclosed by S_4 is zero, ϕ_E is zero.

28.6. Describe a sphere to represent the Gaussian surface enclosing the mass *m*. The angle between the field direction and the element of area on the sphere is 180°. Thus

$$\frac{1}{4\pi G} \phi_g = \frac{1}{4\pi G} \oint g.d\mathbf{S} = -\frac{1}{4\pi G} \int g dS$$

$$= -\frac{g}{4\pi G} \int dS = -\frac{g\, 4\pi r^2}{4\pi G}$$

$$= -\frac{gr^2}{G} = m$$

or $$g = -\frac{Gm}{r^2}$$

A mass *M* placed in the gravitational field *g* due to *m* would experience a force given by

$$F = gM = -\frac{GmM}{r^2}$$

The negative sign signifies that the force is attractive.

28.7. Electric field at point *a* at distance *r* from the charge is given by

$$E = \frac{q}{4\pi\varepsilon_0 r^2} = \frac{(1.0\times10^{-7}\text{ coul})\,(9\times10^{9}\text{ nt-m}^2/\text{coul}^2)}{(1.5\times10^{-2}\text{ meter})^2}$$

$$= 4\times10^{6}\text{ nt/coul}$$

4×10^{6}

At the point *b*, the electric field $E=0$ since the point is within the conductor.

28.8. (*a*) Consider a point charge *q* located at the center of an uncharged thin metallic surface (shell) of radius *R*. Let the point *P* lie on an element of surface *ds* within the shell, Fig. 28. 8 (*a*), on the surface of a sphere of radius *r* concentric with the shell. Since the net charge enclosed within this Gaussian surface is *q*, accordingly to the Gauss' theorem, the flux at *P* is given by

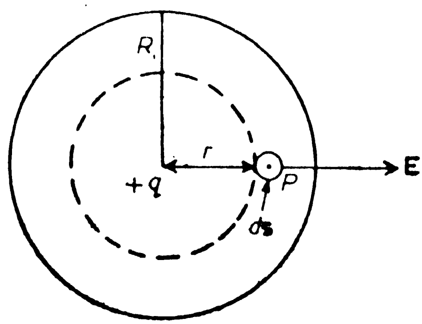


Fig 28.8 (a)

$$\phi_E = \frac{q}{\varepsilon_0} = \oint \mathbf{E}.d\mathbf{S}$$

$$= \int E ds \cos\theta$$

But here the entire field is normal to the surface and points out from it, i.e. $\theta=0$.

$$\therefore \quad \frac{q}{\varepsilon_0} = E\int ds = (E)\,(4\pi r^2)$$

$$E = \frac{q}{4\pi\varepsilon_0 r^2}$$

(*b*) As the shell is conductor, negative charge $-q$ will be induced on the inside and $+q$ will be induced on the outside of the shell. Choose a point P outside the shell at distance r from the center of the shell and draw a spherical Gaussian surface of radius r and concentric with the conducting shell. The surface is indicated in Fig. 28.8 (*b*) by the dotted lines.

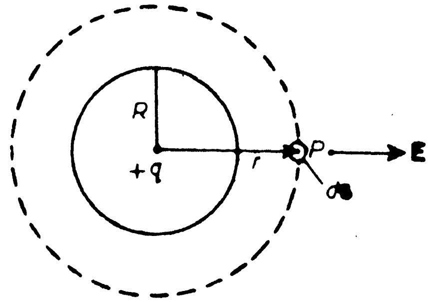


Fig 28.8 (b)

The field vector emerges everywhere normal to the Gaussian surface. Further, the field has constant value over the surface. By Gauss' theorem

$$\varepsilon_0 \oint \mathbf{E}\cdot d\mathbf{S}=q$$

$$\varepsilon_0 E \int ds=q$$

$$\varepsilon_0 E\,(4\pi r^2)=q$$

whence $$E=\frac{q}{4\pi\varepsilon_0 r^2}$$

(*c*) It is seen from the results of (*a*) and (*b*) that the shell has no effect on the field.

(*d*) Yes, negative charge is induced on the inside surface and positive charge on the outside surface of the shell.

(*e*) Yes

(*f*) No

(*g*) No

28.9. The small rectangle in Fig. 28.9 (*a*) is a side view of a closed surface, shaped like a pillbox. Its ends, of area dA, are perpendicular to the figure, one of them lying within the sheet the other in the field. Lines of force crossing the surface of the pillbox is EdA where E is the electric intensity. The charge within the pillbox is σdA. Then from Gauss' law

$$E\,dA=\frac{1}{\varepsilon_0}\ \sigma dA$$

or $$E=\frac{\sigma}{\varepsilon_0}$$

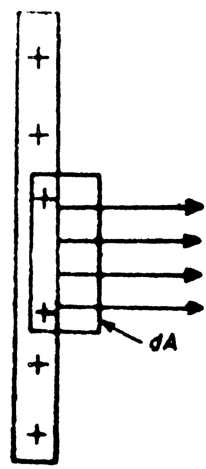


Fig 28.9 (a)

(*a*) On the left of the sheets, Fig 28.9 (*b*) electric intensity E_1 due to sheet 1 of charge on the left hand is directed toward the left and its magnitude is $\sigma/2\varepsilon_0$. Then intensity E_2 due to sheet 2 of charge on the right hand side is also toward the left and its magnitude is also $\sigma/2\varepsilon_0$.

The resultant intensity is therefore,

$$E=E_1+E_2=\frac{\sigma}{2\varepsilon_0}+\frac{\sigma}{2\varepsilon_0}=\frac{\sigma}{\varepsilon_0}$$

(*b*) Between the sheets E_1 and E_2 are in opposite directions and their resultant is zero.

(*c*) On the right hand side of the sheets, E_1 and E_2 again add up and the magnitude of the resultant is σ/ε_0, directed towards right.

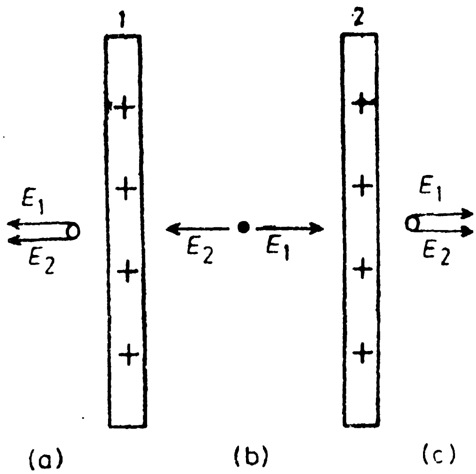


Fig. 28.9 (b)

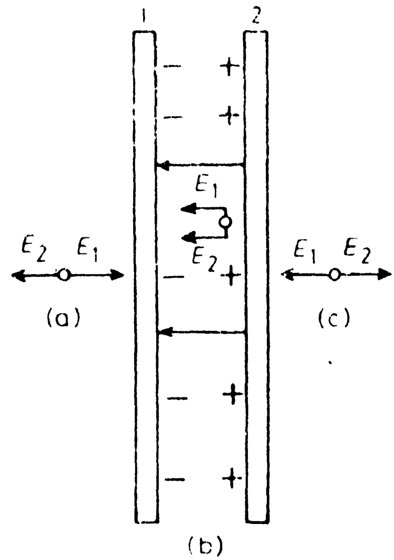


Fig. 28.10

28.10. (*a*) and (*c*). On the left side as well as on the right side of the sheets the intensity components E_1 and E_2 are each of magnitude $\sigma/2\varepsilon_0$ but are oppositely directed so that their resultant is zero (Fig 28.10).

(*b*) At any point between the plates the field components are in the same direction and their resultant is σ/ε_0 and is directed towards left.

28.11. By Problem 28.10, the electric field intensity E between the plates is given by

$$E=\sigma/\varepsilon_0$$

But, the charge density $\sigma=q/A$

$$\therefore \quad q=E\varepsilon_0 A$$

$$=(55\ \text{nt/coul})(8.9\times10^{-12}\ \text{coul}^2/\text{nt-m}^2)\ (1.0\ \text{meter}^2)$$

$$=4.9\times10^{-10}\ \text{coul.}$$

28.12. (*a*) As no charge is enclosed within the charged metal sphere, E for a point inside the sphere is zero.

(*b*) In calculating the field external to the spherical charge distribution, the charge may be considered as concentrated at the center.

$$E=\frac{q}{4\pi\varepsilon_o r^2}=\frac{(9\times10^9 \text{ nt-m}^2/\text{coul}^2)\,(2\times10^{-7}\text{ coul})}{(0.25\text{ meter})^2}$$

$$=2.9\times10^4 \text{ nt/coul.}$$

(*c*) $$E=\frac{q}{4\pi\varepsilon_o r^2}=\frac{(9\times10^9 \text{ nt-m}^2/\text{coul}^2)\,(2\times10^{-7}\text{ coul})}{(3\text{ meter})^2}$$

$$=200 \text{ nt/coul.}$$

28.13. $E=\dfrac{\sigma}{\varepsilon_o}$

Force on the electron, $F=Ee=\dfrac{\sigma e}{\varepsilon_o}$

Where e is the electron charge.

Let the electron be fired from a distance x meters so as to just miss striking the plate. Then work done

$$W=Fx=\frac{\sigma e x}{\varepsilon_o}$$

Setting, $W=K$, the initial electron kinetic energy

$$\frac{\sigma e x}{\varepsilon_o}=K$$

$$x=\frac{K\varepsilon_o}{\sigma e}$$

$$=\frac{(100\text{ ev})\,(1.6\times10^{-19}\text{ joule/ev})\,(8.9\times10^{-12}\text{ coul}^2/\text{nt-m}^2)}{(-2\times10^{-6}\text{ coul/meter}^2)\,(-1.6\times10^{-19}\text{ coul})}$$

$$=0.44\times10^{-3}\text{ meter}=0.44\text{ mm.}$$

28.14. Describe a Gaussian surface in the form of a right cylinder of radius r coaxial with the given cylinder and of length b Fig 28.14 (*b*). Let $r < R$. The charge within this Gaussian surface is

$$q=\pi r^2 b\,\rho$$

As the caps of the cylinder are perpendicular to **E**, they do not contribute to the field intensity. The only surface which is of consequence is that of the cylinder's curved surface of area

$$A=2\pi r b,$$

According to Gauss' law

$$\varepsilon_o \oint \mathbf{E}.d\mathbf{S}=q$$

As **E** is normal to the elementary surface d**S** and is constant, we have

$$\varepsilon_o E \int ds=q$$

$$\varepsilon_o EA=q$$

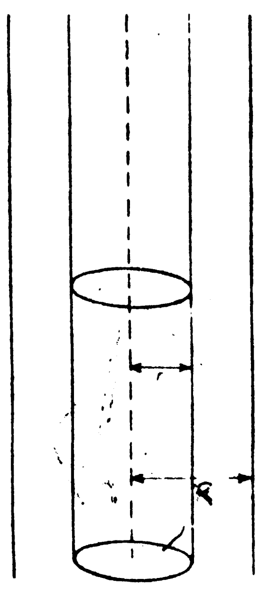


Fig. 28.14 (a)

$$\therefore \qquad E=\frac{q}{\varepsilon_o A}=\frac{\pi r^2 b\rho}{\varepsilon_o(2\pi rb)}=\frac{\rho r}{2\varepsilon_o}\ (r<R)$$

For $r > R$, again construct the closed Gaussian surface in the form of a right cylinder of radius r and length b coaxial with the given cylinder.

No lines of force cross the ends of the cylinder. The lines of force cross outward normal, to the curved surface as before. We have

$$q=\pi R^2 b\rho$$
$$A=2\pi rb$$
$$E=\frac{q}{\varepsilon_o A}=\frac{\pi R^2 b\rho}{\varepsilon_o(2\pi rb)}=\frac{R^2\rho}{2\varepsilon_o r}$$

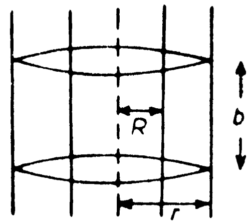


Fig. 28.14 (b)

28.15. For $r < a$; $E=0$

For $a < r < b$; $E=\frac{q'}{4\pi\varepsilon_o r^2}=\frac{4}{3}\,\frac{\pi(r^3-a^3)\rho}{4\pi\varepsilon_o r^2}=\frac{(r^3-a^3)\rho}{3\varepsilon_o r^2}$

For $r > b$; $E=\frac{q}{4\pi\varepsilon_o r^2}=\frac{4}{3}\,\frac{\pi(b^3-a^3)\rho}{4\pi\varepsilon_o r^2}=\frac{(b^3-a^3)\rho}{3\varepsilon_o r^2}$

Fig. 28.15 shows the plot of E versus r.

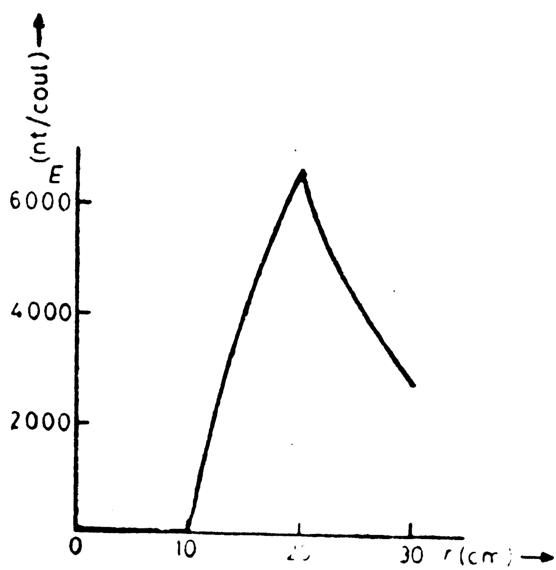


Fig. 28.15

28.16. Case (i) $r > R$.

Construct a Gaussian surface in the form of a right cylinder of radius r and of length b, coaxial with the metal tube. As the lines of force do not cross the ends of the cylinder, the only surface that matters is the curved surface of the cylinder through which lines of force cross in an outward direction normal to the surface. The quantity of charge within cylinder is λb. According to Guass' theorem.

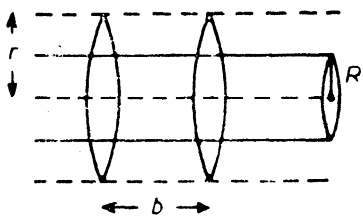


Fig. 28.16 (a)

$$(\varepsilon_0 E)(2\pi r b) = \lambda b$$

or
$$E = \frac{\lambda}{2\pi\varepsilon_0 r} \quad (r > R)$$

Case (ii) $r < R$.

Since no charge resides within the tube, the field

$$E = 0 \quad (r < R)$$

Fig. 28.16 (b) shows the plot of E versus r.

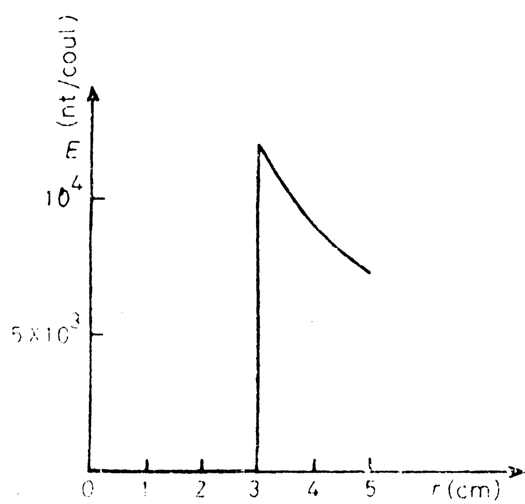


Fig. 28.16 (b)

28.17. (a) For $r > b$ the point is outside both the cylinders and the Gaussian surface drawn at radial distance r would enclose a net charge equal to zero since the two cylinders carry equal and opposite charge. Hence, $E=0$

Again, for $r < a$, the Gaussian surface does not enclose any charge. Hence $E=0$.

(b) Between the cylinders, we have $a < r < b$. Describe a Gaussian surface in the form of a right cylinder of radius r and length L coaxial with the given cylinders. Then the charge enclosed by the

Gaussian surface is λL and the curved surface through which E is projected outward and is normal to the surface has area $A=2\pi rL$. The ends of the cylinder do not contribute to the field intensity. By Gauss' theorem

$$(\varepsilon_o E)\ (2\pi rL)=\lambda L$$

or
$$E=\frac{\lambda}{2\pi\varepsilon_o r}$$

28.18. $E=\dfrac{\lambda e}{2\pi\varepsilon_o r}$

Electric force acting on the positron

$$F=Ee=\frac{\lambda e}{2\pi\varepsilon_o r}$$

Equating the electric force to the centripetal force

$$\frac{\lambda e}{2\pi\varepsilon_o r}=\frac{mv^2}{r}$$

Whence, the kinetic energy of positron

$$K=\tfrac{1}{2}mv^2=\frac{\lambda e}{4\pi\varepsilon_o}$$

$$=(9\times 10^9 \text{ nt-m}^2/\text{coul}^2)\ (3\times 10^{-8} \text{ coul/meter})\ (1.6\times 10^{-19} \text{ coul})$$

$$=432\times 10^{-19} \text{ joules}$$

$$=(432\times 10^{-19} \text{ joules})/(1.6\times 10^{-19} \text{ joules/ev})=270 \text{ ev.}$$

28.19. (*a*) Describe a Gaussian surface in the form of a right cylinder of radius r and length b. The area of the curved surface which alone contributes to the field intensity is given by $A=2\pi rb$. The net charge enclosed by the Gaussian surface is $-2q+q$ or $-q$.

By Gauss's law

$$(\varepsilon_o E)\ (2\pi rb)=-q$$

whence

$$E=-\frac{q}{2\pi\varepsilon_o br}$$

The negative sign shows that E is directed inward.

(*b*) A charge $-q$ will be distributed on the inside surface of the shell and a charge $-q$ will reside on the outside surface.

(*c*) Between the cylinders, the Gaussian surface (again cylindrical in shape) will enclose a net charge $+q$ so that

$$E=\frac{q}{2\pi\varepsilon_o br}$$

The field being radially outward

Assumptions made are

(*i*) The cylinder is sufficiently long so that only radial component of the field exists.

(*ii*) Fringing field near the ends of the cylinder is not present.

(*iii*) The charge is uniformly distributed.

28.20. (*a*) Construct a Gaussian surface in the form of a sphere of radius r, concentric with the spherical shells. Since no charge is enclosed by the Gaussian surface with $r < a$, $E=0$.

(*b*) Here the net charge enclosed by the Gaussian surface is q_a, As E is normal to the spherical surface by Gauss's law

$$(\varepsilon_o E)\,(4\pi r^2)=q_a \qquad \text{or} \qquad E=\frac{q_a}{4\pi\varepsilon_o r^2}$$

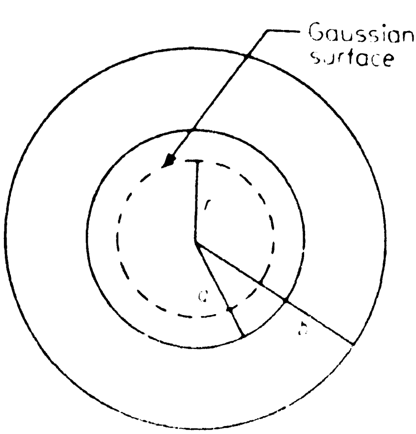


Fig 28.20 (a)

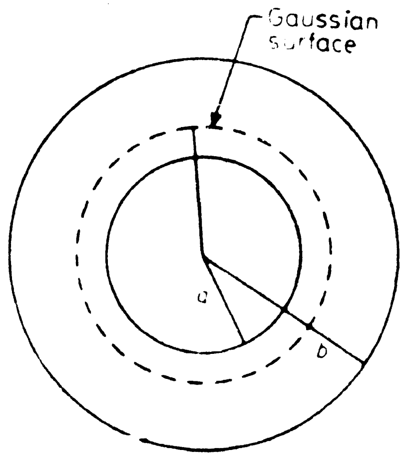


Fig 28.20 (b)

(*c*) Here the net charge enclosed by the Gaussian surface is $q_a + q_b$, and E is normal to the spherical surface. By Gauss's law

$$(\varepsilon_o\, E)\,(4\pi r^2)= q_a + q_b. \qquad \text{or} \qquad E=\frac{q_a+q_b}{4\pi\varepsilon_o\, r^2}.$$

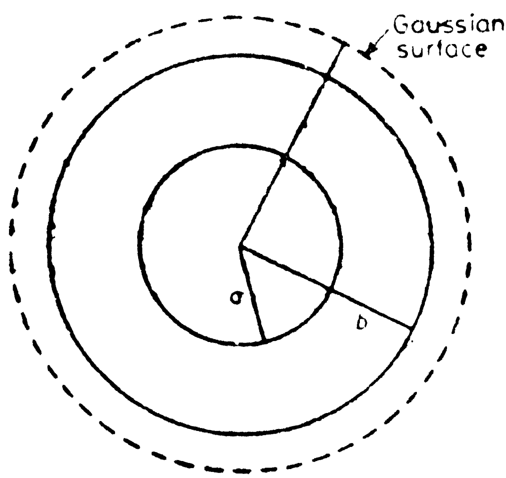


Fig 28.20 (c)

28.21. For the conducting sheet the field is given by

$$E=\frac{\sigma}{\varepsilon_0} \qquad \ldots(1)$$

The electric force acting on the sphere is

$$F=Eq=\frac{\sigma q}{\varepsilon_0} \qquad \ldots(2)$$

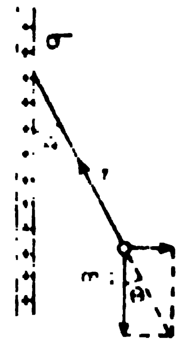


Fig. 28.21

The sphere is held in equilibrium under the joint action of three force: (*i*) weight mg acting down, (*ii*) electric force F acting horizontally, and (*iii*) tension in the thread acting at an angle θ with the vertical.

We have from Fig. 28.21

$$\frac{F}{mg}=\tan\theta \qquad \ldots(3)$$

Using (2) and (3)

$$\sigma=\frac{\varepsilon_0 mg\tan\theta}{q}$$

$$=\frac{(8.9\times10^{-12}\ \text{coul}^2/\text{nt-m}^2)(1.0\times10^{-6}\ \text{kg})(9.8\ \text{m/s}^2)\tan 30^\circ}{(2.0\times10^{-8}\ \text{coul})}$$

$$=2.5\times10^{-9}\ \text{coul/meter}^2.$$

28.22. $E=\sigma/\varepsilon_0$.

For a sphere $\sigma=q/4\pi r^2$, since the surface area of a sphere is $4\pi r^2$.

$$\therefore \qquad E=\frac{q}{4\pi\varepsilon_0 r^2}$$

28.23. The α-particle is at a distance $r=2R$ from the center of the gold nucleus of radius R $(=6.9\times10^{-15}$ meter). Considering α-particle to be a point charge, the electric force is

$$F=\frac{Z_1Z_2e^2}{4\pi\varepsilon_0 r^2}$$

where Z_1e and Z_2e are the charges of the α-particle and the gold nucleus respectively.

$$F=\frac{(2\times1.6\times10^{-19}\ \text{coul})(79\times1.6\times10^{-19}\ \text{coul})(9\times10^{9}\ \text{nt-m}^2/\text{coul}^2)}{(2\times6.9\times10^{-15}\ \text{meter})^2}$$

$=191$ nt.

$$\text{Acceleration, } a=\frac{F}{m}=\frac{191\ \text{nt}}{6.7\times10^{-27}\ \text{kg}}=2.85\times10^{28}\ \text{meter/sec}^2$$

28.24. (*a*) Consider 1 cm² of face area. Then the volume of the gold foil, 3×10^{-5} cm thick, is

$$v=3\times10^{-5}\ \text{cm}\times1\ \text{cm}\times1\ \text{cm}=3\times10^{-5}\ \text{cm}^3$$

Number of gold nuclei per cm^3 is

$$N=\frac{N_0\rho}{A}$$

where N_0 is the Avogadro's number, ρ the density and A the atomic weight of the material.

$$N=(6\times10^{23}\text{ atoms/gm atom})\,(19.3\text{ gm/cm}^3)/197$$
$$=5.88\times10^{22}\text{ atoms/cm}^3$$

Number of gold atoms in volume v is

$$n=Nv=(5.88\times10^{22}\text{ atoms/cm}^3)\,(3\times10^{-5}\text{ cm}^3)$$
$$=1.76\times10^{18}$$

If σ is the area of each gold nucleus, then the total area arising from n nuclei is

$$S=n\sigma$$

If R is the radius of gold nucleus, then

$$\sigma=\pi R^2=\pi(6.9\times10^{-13}\text{ cm})^2=1.5\times10^{-24}\text{ cm}^2$$
$$S=(1.76\times10^{18})\,(1.5\times10^{-24}\text{ cm}^2)=2.6\times10^{-6}\text{ cm}^2$$

$\therefore$ Fraction of surface area "blocked out" by gold nuclei

$$=\frac{2.6\times10^{-6}\text{ cm}^2}{1.0\text{ cm}^2}=2.6\times10^{-6}$$

(*b*) Volume occupied by each gold nucleus

$$v=\frac{4}{3}\pi R^3=\frac{4}{3}\pi(6.9\times10^{-13}\text{ cm})^3=1.38\times10^{-36}\text{ cm}^3$$

Volume occupied by N nuclei per cm^3 of foil is

$$Nv=(5.88\times10^{22}\text{ atoms/cm}^3)\,(1.38\times10^{-36}\text{ cm}^3)$$
$$=8.1\times10^{-14}\text{ cm}^3$$

Fraction of volume of the foil occupied by the nuclei is

$$\frac{8.1\times10^{-14}\text{ cm}^3}{1.0\text{ cm}^3}=8.1\times10^{-14}$$

(*c*) Rest of the space is filled with electrons. But, a major part of the space remains empty.

28.25. (*a*) The flux is completely determined by the x-component of the field as $E_y=E_z=0$.

In-flux. $\phi_{in}=a^2E_x=a^2b\sqrt{a}=ba^{5/2}$

Out-flux. $\phi_{out}=a^2b\sqrt{2a}=\sqrt{2}\,ba^{5/2}$

Net outward flux. $\phi=\phi_{out}-\phi_{in}$

$$=\sqrt{2}\,ba^{5/2}-ba^{5/2}$$
$$=(\sqrt{2}-1)\,ba^{5/2}$$

$$=(\sqrt{2}-1)\ (800\ \text{nt/coul-m}^{\frac{1}{2}})\ (0.1\ \text{meter})^{5/2}$$
$$=1.05\ \text{nt-meter}^2/\text{coul}$$

(*b*) $$q=\varepsilon_0\phi$$
$$=(8{:}9\times10^{-12}\ \text{coul}^2/\text{nt-m}^2)\ (1.05\ \text{nt-m}^2/\text{coul})$$
$$=9.3\times10^{-12}\ \text{coul}$$

SUPPLEMENTARY PROBLEMS

S.28.1. Consider a cube of side $a=100$ meter which encloses a charge q, the upper and lower surfaces of the cube being at 300 meter and 200 meter altitude.

By Gauss theorem the flux is given by

$$\phi_E=\frac{q}{\varepsilon_0}=\int \mathbf{E}.d\mathbf{S}$$

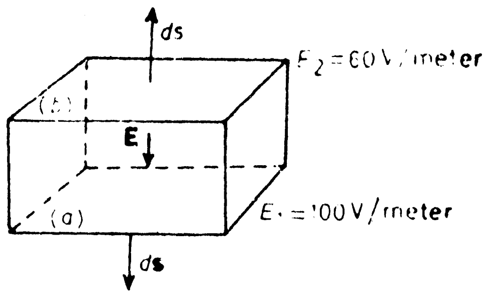


Fig S.28.1

The flux can be written as the sum of six terms; (*a*) integral over the bottom surface (*b*) integral over the top surface and (*c*) integral over the four vertical faces.

$$\phi_E=\int_{(a)} \mathbf{E}_1.d\mathbf{S}+\int_{(b)} \mathbf{E}_2.\ d\mathbf{S}+4\int_{(c)} \mathbf{E}.d\mathbf{S}$$

In (*a*), E and *d*S point in the same direction so that the angle θ between these to vectors is zero.

$$\int_{(a)} \mathbf{E}_1.d\mathbf{S}=\int E_1\cos 0^\circ\ dS=E_1\int dS=E_1S$$

where $S=a^2$, is the area of the face.

In (*b*), $\mathbf{E}_2$ and *d*S point in the opposite direction so that $\theta=180^\circ$.

$$\int_{(b)} \mathbf{E}_2.d\mathbf{S}=\int E_2\cos 180^\circ\ dS=-E^2\int dS=-E_2S$$

In (*c*), E and *d*S point at right angles so that $\theta=90^\circ$. For each vertical face,

$$\int_{(a)} \mathbf{E}.d\mathbf{S}=\int E\cos 90^\circ\ dS=0.$$

$$\phi_E=E_1S-E_2S=(E_1-E_2)S=(E_1-E_2)\ a^2=\frac{q}{\varepsilon_0}$$

$$q = \varepsilon_0 (E_1 - E_2)\, a^2$$

$$= (8.85 \times 10^{-12}\ \text{coul}^2/\text{nt-m}^2)\left(100\frac{\text{volt}}{\text{meter}} - 60\frac{\text{volt}}{\text{meter}}\right)(100\ \text{meter})^2$$

$$= 3.54 \times 10^{-6}\ \text{coul}.$$

S.28.2. Electric field extends from higher potential to a lower one. In order that the electric field may have constant direction the charge in the region must be uniformly distributed in planes perpendicular to E. The fact that field is decreasing in strength in the direction of E implies that the charge is negative.

S.28.3. Potential difference between the concentric spherical shells of radii a and b is

$$V = \frac{-Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$= -(-6 \times 10^{-8}\ \text{coul})\left(9 \times 10^9\ \frac{\text{nt-m}^2}{\text{coul}^2}\right)\left(\frac{1}{0.145\ \text{meter}} - \frac{1}{0.207\ \text{meter}}\right)$$

$$= 1115\ \text{volt}$$

Kinetic energy gained by electron of charge e in falling through a potential difference of V volts is eV

$$\tfrac{1}{2}mv^2 = eV$$

$$v = \sqrt{\frac{2ev}{m}} = \sqrt{\frac{(2)(1.6 \times 10^{-19}\ \text{coul})(1115\ \text{volt})}{(9.1 \times 10^{-31}\text{kg})}}$$

$$= 1.98 \times 10^7\ \text{meter/sec}.$$

S.28.4. (*a*) The electric force between the two spheres would be as if the charge is concentrated at their centers.

By Coulmb's law

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{R^2}$$

where R is the distance between the centers of the spheres.

(*b*) If the charges are like, then force will be less than in (*a*) and for unlike charges the force will be greater than in (*a*).

the two cases must be distinguished.

(*i*) The two spheres have like charges.

In this case owing to coulumb's repulsion, the charges on each sphere instead of remaining uniformly distributed with the center of charges coinciding with the centers of the spheres are now displaced towards the rear surfaces of the spheres, resulting in a greater value for R, the distance of separation of the centers of charges. This has the consequence of reducing the force.

(*ii*) The two spheres have unlike charges.

In this case the coulomb's attraction causes the charges to be pulled towards the front surface of the spheres, leading to a reduction in the effective value of R. This has the consequence of increasing the force.

S.28.5. The field due to the point charge Q at the center at a distance r is

$$E_1 = \frac{Q}{4\pi\varepsilon_o r^2}$$

The charge between the spheres of radii r and a is

$$q = \int_a^r \rho(4\pi r^2)\, dr = \int_a^r \frac{A}{r}(4\pi r^2)\, dr = 4\pi A \int_a^r r\, dr$$

$$= 2\pi A(r^2 - a^2)$$

By Gauss theorem

$$\varepsilon_o \int \mathbf{E}.d\mathbf{S} = 4\pi r^2\, \varepsilon_o E_2 = q = 2\pi A\,(r^2 - a^2)$$

$$\therefore \qquad E_2 = \frac{A}{2\varepsilon_o r^2}(r^2 - a^2)$$

Total field, $E = E_1 + E_2 = \dfrac{Q}{4\pi\varepsilon_o r^2} + \dfrac{A}{2\varepsilon_o r^2}(r^2 - a^2)$

$$= \frac{A}{2\varepsilon_o} + \frac{Q - 2\pi A a^2}{4\pi\varepsilon_o r^2}$$

If E is to remain constant in the region, $a < r < b$, for any value of r, then the numerator of the second term of the right side in the above expression must vanish.

$$Q - 2\pi A a^2 = 0$$

Hence, $\qquad A = \dfrac{Q}{2\pi a^2}$

S.28.6. (a) Consider a Gaussian surface of radius r.

By Gauss' Theorem,

R. ID. No.

$$\varepsilon_o \oint \mathbf{E}.d\mathbf{S} = q_o$$

$$\mathbf{E} = \mathbf{a}_r \frac{q_o}{4\,\pi\varepsilon_o r^2} \qquad \ldots(1)$$

where $\mathbf{a}_r$ is the unit vector pointing toward P from the center of the sphere, and q_o is the total amount of charge within the Gaussian surface.

For uniform charge density, ρ is constant and

$$q_o = \frac{4}{3}\pi r^3 \rho$$

Using (2) in (1)

$$\mathbf{E}=\frac{\rho \mathbf{a}_r r}{3\,\varepsilon_0}=\frac{\rho \mathbf{r}}{3\,\varepsilon_0}$$

(*b*) The electric field at the center of the cavity due to the remaining portion of the sphere is

$$\mathbf{E}_1=\frac{\rho}{3\,\varepsilon_0}(\mathbf{a}-\mathbf{R})$$

where R is the radius of the cavity. The electric field due to a sphere of radius R corresponding to the volume of the cavity is

$$\mathbf{E}_2=\frac{\rho}{3\,\varepsilon_0}\mathbf{R}$$

Using superposition principle the electric field at any point within the cavity for uniform field is

$$\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2$$

$$=\frac{\rho}{3\,\varepsilon_0}(\mathbf{a}-\mathbf{R})+\frac{\rho}{3\,\varepsilon_0}\mathbf{R}$$

$$=\frac{\rho \mathbf{a}}{3\,\varepsilon_0}$$

29 ELECTRIC POTENTIAL

29.1. The field intensity, $E=\dfrac{\sigma}{2\varepsilon_0}=\dfrac{(1.0\times10^{-7}\ \text{coul/m}^2)}{(2)\ (8.9\times10^{-12}\ \text{coul}^2/\text{nt-m}^2)}$

$$=5{\cdot}6\times10^3\ \text{nt/coul}$$

Let the equipotentials be Δs meters apart. As E is constant,

$$E\Delta s=\Delta V$$

or $$\Delta s=\frac{\Delta V}{E}=\frac{5.0\ \text{volt}}{5.6\times10^3\ \text{nt/coul}}=0.89\times10^{-3}\ \text{meter}$$

$$=0.89\ \text{mm.}$$

29.2. (*a*) The potential V_B at the point B is given by

$$V_B=V_A-\int_{r_A}^{r_B}Edr$$

Setting $r_A=\infty$; $V_A=0$; $r_B=a$

$$V_B=V_a$$

$$V_a=-\int_{\infty}^{a}Edr$$

$$=-\int_{R}^{a}Edr-\int_{\infty}^{R}Edr$$

For the first integral, we find E using Gauss' theorem

$$4\ \pi r^2\ \varepsilon_0 E=q'$$

where q' is the charge enclosed within a radius r. Since q' is proportional to the volume we have

$$q'=\frac{r^3}{R^3}q$$

$$\therefore\qquad E=\frac{r^3q}{4\pi r^2\varepsilon_0 R^3}=\frac{qr}{4\pi\varepsilon_0 R^3}$$

The first integral is then evaluated as follows:

$$-\int_{R}^{a}Edr=\int_{a}^{R}\frac{qr\,dr}{4\ \pi\varepsilon_0 R^3}=\frac{q\,(R^2-a^2)}{8\ \pi\varepsilon_0 R^3}$$

For the second integral, Gauss' theorem gives

$$4\pi r^2\varepsilon_0 E=q$$

$$E=\frac{q}{4\pi\varepsilon_0 r^2}$$

and the integral becomes

$$-\int_{\infty}^{R} E dr=-\int_{\infty}^{R}\frac{q\,dr}{4\pi\varepsilon_0 r^2}=\frac{q}{4\pi\varepsilon_0 R}$$

$$\therefore \quad V_a=\frac{q}{8\pi\varepsilon_0 R^3}\left(R^2-a^2\right)+\frac{q}{4\pi\varepsilon_0 R}=\frac{q}{8\pi\varepsilon_0 R^3}\left(3R^2-a^2\right)$$

(*b*) Yes. Actually, $V=0$ at ∞.

29.3. The potential is given by

$$V=\frac{q}{4\pi\varepsilon_0 R}=\frac{(10^{-8}\text{ coul})\,(9\times10^9\text{ nt-m}^2/\text{coul}^2)}{(0.1\text{ meter})}=900\text{ volt.}$$

29.4. (*a*) $R=\dfrac{q}{4\pi\varepsilon_0 V}=\dfrac{(1.5\times10^{-8}\text{ coul})\,(9\times10^9\text{ nt-m}^2/\text{coul}^2)}{(30\text{ volt})}$

$=4.5$ meter.

(*b*) $$V=\frac{q}{4\pi\varepsilon_0 R}$$

$$\therefore \quad \Delta V=-\frac{q}{4\pi\varepsilon_0}\frac{\Delta R}{R^2}$$

or $$\Delta R=-\frac{4\pi\varepsilon_0\,\Delta V R^2}{q}$$

Clearly, ΔR depends on R. Hence **surfaces** whose potentials differ by a constant amount are not evenly spaced.

29.5. (*a*) (*i*) Let $V=0$ at distance x from $+q$, between the charges, then,

$$\frac{q}{4\pi\varepsilon_0 x}-\frac{3q}{4\pi\varepsilon_0(d-x)}=0$$

$$d-x-3x=0$$

$$x=\frac{d}{4}=\frac{100\text{ cm}}{4}=25\text{ cm}$$

(*ii*) Let $V=0$ at distance x from $+q$ outside the charges. Then

$$\frac{q}{4\pi\varepsilon_0 x}-\frac{3q}{4\pi\varepsilon_0(d+x)}=0$$

$$d+x-3x=0$$

or $\qquad x=\frac{d}{2}=\frac{100\text{ cm}}{2}=50\text{ cm}.$

(*b*) Let $E=0$ at distance x from $+q$ outside the charges. Then

$$\frac{q}{4\pi\varepsilon_0 x^2}-\frac{3q}{4\pi\varepsilon_0 (d+x)^2}=0$$

$\therefore \qquad (d+x)^2-3x^2=0$

or $\qquad 2x^2-2dx-d^2=0$

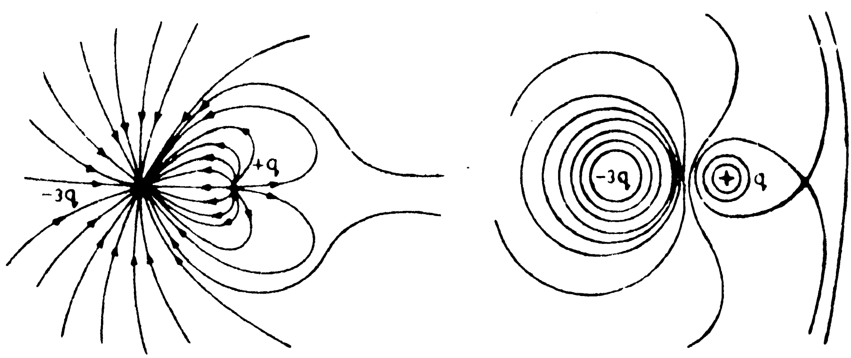
Set $\qquad d=1.0$ meter. Then

$$2x^2-2x-1=0$$

$$x=\frac{1+\sqrt{3}}{2}=1.37\text{ meter}=137\text{ cm}.$$

Between the charges E cannot be zero since the forces due to $+q$ and $-3q$ would act in the same direction.

29.6.



(a) (b)

29.7. $\qquad V_A=\frac{q}{4\pi\varepsilon_0 a}-\frac{q}{4\pi\varepsilon_0 (a+d)}=\frac{qd}{4\pi\varepsilon_0 a(a+d)}$

$$V_B=-\frac{q}{4\pi\varepsilon_0 a}+\frac{q}{4\pi\varepsilon_0 (a+d)}=-\frac{qd}{4\pi\varepsilon_0 a(a+d)}$$

$$\therefore V_A-V_B=\frac{qd}{4\pi\varepsilon_0 a(a+d)}-\frac{-qd}{4\pi\varepsilon_0 a(a+d)}=\frac{qd}{2\pi\varepsilon_0 a(a+d)}$$

When $d=0$, A and B coincide and therefore, $V_A-V_B\rightarrow V_A-V_A=0$, which is the expected result. Also, when $q=0$, $V_A=V_B=0$.

Hence, $V_A-V_B=0$ is again an expected result.

29.8. (*a*) $V_A=\frac{q}{4\pi\varepsilon_0 r_A}$

$$V_B=\frac{q}{4\pi\varepsilon_0 r_B}$$

where r_A and r_B are respectively distances of A and B from q.

$$V_A - V_B = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

$$= (9\times10^9 \text{ nt-m}^2/\text{coul}^2)(1.0\times10^{-6}\text{coul})\left(\frac{1}{2.0 \text{ meter}} - \frac{1}{1.0 \text{ meter}}\right)$$

$$= -4500 \text{ volts}$$

(b) $V_A - V_B = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) = -4500$ volts

29.9. The center of negative charge lies at O, the oxygen nucleus. On the other hand the center of positive charge of the hydrogen atoms lies at P, midway between the two protons. The distance of separation of the center of positive charge and center of negative charge denoted by a can be calculated from the triangle OPH

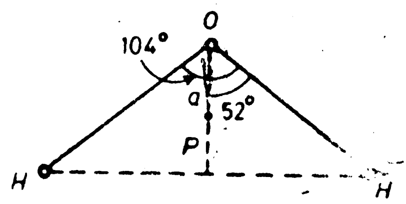


Fig. 29.9

$$a = (OH)\cos 52°$$
$$= (0.96\times10^{-10} \text{ meter})\,(0.616) = 0.59\times10^{-10} \text{ meter}$$

Dipole moment arises due to the separation of $+2e$ and $-2e$ charges of hydrogen atoms by distance a.

Dipole moment $p = (2\,e)\,(a)$

$$= 2\,(1.6\times10^{-19}\text{coul})\,(0.59\times10^{-10} \text{ meter}) = 1.9\times10^{-29} \text{ coul-meter.}$$

This value is to be compared with the figure of 0.6×10^{-29} coul-meter quoted in the text which is lower but correct. The discrepency is to be attributed to the oversimplified model.

29.10. Potential at P is given by the sum of potentials due to the charges $+q$, $+q$ and $-q$ at distance $(r-a)$, r and $(r+a)$ respectively.

$$V = \frac{q}{4\pi\varepsilon_0(r-a)} + \frac{q}{4\pi\varepsilon_0 r} - \frac{q}{4\pi\varepsilon_0(r+a)}$$

$$\therefore \quad V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{2qa}{r^2 - a^2}\right)$$

Since $r \gg a$, $r^2 - a^2 \simeq r^2$

$$\therefore \quad V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{2qa}{r^2}\right)$$

Right hand side is nothing but the potential arising due to an isolated charge plus a dipole at distance r.

29.11. Energy released is

$$K = qV = (30 \text{ coul})\ (10^9 \text{ volts})$$
$$= 3 \times 10^{10} \text{ joules}$$
$$= (3 \times 10^{10} \text{ joules})/(4.18 \text{ joules/cal})$$
$$= 7.2 \times 10^9 \text{ cal}$$

Heat required to melt m gms of ice at 0° C is mL, where L is the latent heat of ice.

$$mL = 7.2 \times 10^9 \text{ cal}$$

$$m = \frac{7.2 \times 10^9 \text{ cal}}{80 \text{ cal/gm}} = 9 \times 10^7 \text{ gm}$$

$$= 9 \times 10^4 \text{ kg} = \frac{9 \times 10^4 \text{ kg}}{10^3 \text{ kg/ton}} = 90 \text{ tons.}$$

29.12. (*a*) The electric potential is

$$V = \frac{q}{4\pi\varepsilon_0 r} = \frac{(9 \times 10^9 \text{ nt-m}^2/\text{coul}^2)\ (1.6 \times 10^{-19} \text{ coul})}{(5.3 \times 10^{-11} \text{ meter})}$$
$$= 27.1 \text{ volts}$$

(*b*) The electric potential energy of the atom, is

$$U = -eV = -27.1 \text{ eV}$$

(*c*) Equating centripetal force to the electrostatic force

$$\frac{mv^2}{r} = \frac{e^2}{4\pi\varepsilon_0 r^2}$$

or $$mv^2 = \frac{e^2}{4\pi\varepsilon_0 r} = 27.1 \text{ ev}$$

Kinetic energy $= \frac{1}{2}\, mv^2$, $\frac{1}{2}$ (27.1 ev) $=$ 13.6 ev

(*d*) Total energy = kinetic energy + potential energy

$$= 13.6 \text{ ev} - 27.1 \text{ ev}$$
$$= -13.5 \text{ ev}$$

Hence, energy required to ionize the hydrogen atom is 13.5 ev.

29.13. The electric potential energy of the charge configuration of the textbook Fig. 29.7 is

$$U = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1q_2}{a} + \frac{q_1q_3}{\sqrt{2}a} + \frac{q_1q_4}{a} + \frac{q_2q_3}{a} + \frac{q_2q_4}{\sqrt{2}a} + \frac{q_3q_4}{a}\right]$$

By Problem, $q_1 = +1.0 \times 10^{-8}$ coul ; $q^2 = -2.0 \times 10^{-8}$ coul;

$q_3 = +3.0 \times 10^{-8}$ coul; $q_4 = +2.0 \times 10^{-8}$ coul ;

and $a = 1.0$ meter

$$U=\frac{(9\times 10^9 \text{ nt-m}^2/\text{coul}^2)}{(10 \text{ meter})}\left[(1)(-2)+\frac{(1)(3)}{\sqrt{2}}\right.$$
$$\left.+(1)(2)+(-2)(3)+\frac{(-2)(2)}{\sqrt{2}}+(3)(2)\right]\times 10^{-16}$$
$$=-6.4\times 10^{-7} \text{ joules.}$$

29.14. (*a*) $V=\dfrac{q}{4\pi\varepsilon_o r}$ ...(1)

$$r=\frac{q}{4\pi\varepsilon_o V}=\frac{(9\times 10^9 \text{ nt-m}^2/\text{coul}^2)(3\times 10^{-6} \text{ coul})}{(500 \text{ volt})}$$
$$=54 \text{ meters}$$

(*b*) Assuming that the drops are incompressible, the volume of new drop will be twice that of the small drop.

$$\frac{4}{3}\pi R^3=2(\frac{4}{3}\pi r^3)$$

∴ Radius of new drop, $R=2^{1/3}\, r.$ ...(2)

The charge of the new drop,

$$Q=2q \qquad ...(3)$$

The potential at the surface of new drop is

$$V_o=\frac{Q}{4\pi\varepsilon_o R}=\frac{2q}{4\pi\varepsilon_o 2^{1/3} r}=\frac{2^{2/3}q}{4\pi\varepsilon_o r}$$
$$=2^{2/3}V \qquad ...(4)$$

where we have used (1), (2) and (3).

$$V_o=2^{2/3}\,(500 \text{ volts})=794 \text{ volts.}$$

29.15. (*a*) Total charge on earth's surface

$$q=(-e)(4\pi r^2)$$

where $-e$ is electron charge and r is earth's radius. The potential is

$$V=\frac{q}{4\pi\varepsilon_o r}=-\frac{4\pi r^2 e}{4\pi\varepsilon_o r}=-\frac{er}{\varepsilon_o}$$
$$=-\frac{(1.6\times 10^{-19} \text{ coul})(6.4\times 10^6 \text{ meter})}{(8.9\times 10^{-12} \text{ coul}^2/\text{nt-m}^2)}=-0.115 \text{ volt.}$$

(*b*) The electric field due to the earth just outside its surface is

$$E=\frac{q}{4\pi\varepsilon_o r^2}=-\frac{4\pi r^2 e}{4\pi\varepsilon_o r^2}=-\frac{e}{\varepsilon_o}$$
$$=-(1.6\times 10^{-19} \text{ coul }(8.9\times 10^{-12} \text{ coul}^2/\text{nt-m}^2)$$
$$=-1.8\times 10^{-8} \text{ nt-m}^2/\text{coul.}$$

The negative sign shows that the electric field points radially inward.

29.16. Under the assumption of constant density the volume of each fragment is half of the U^{238} nucleus.

$$\frac{4}{3}\pi r^3 = \frac{1}{2}\left(\frac{4}{3}\pi R^3\right)$$

$\therefore$ Radius of each fragment, $r = \frac{R}{2^{1/3}} = \frac{8\times10^{-15}\text{ meter}}{2^{1/3}}$

The distance between the centers of the fragments

$$d = 2r = 2(8\times10^{-15}\text{ meter})/2^{1/3} = 1.27\times10^{-14}\text{ meter}$$

The charge on each fragment is $q_1 = q_2 = \frac{Ze}{2}$

(*a*) Force acting on each fragment is

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 d^2} = \frac{(Ze)^2}{4\pi\varepsilon_0 d^2}$$

$$= \frac{(9\times10^9\text{ nt-m}^2/\text{coul}^2)\,(46\times1.6\times10^{-19}\text{ coul})^2}{(1.27\times10^{-14}\text{ meter})^2}$$

$$= 3020\text{ nt.}$$

(*b*) Mutual electric potential energy of the two fragments is

$$U = \frac{(Ze)^2}{4\pi\varepsilon_0 d} = \frac{(9\times10^9\text{ nt-m}^2/\text{coul}^2)\,(46\times1.6\times10^{-19}\text{ coul})^2}{(1.27\times10^{-14}\text{ meter})}$$

$$= 3.8\times10^{-11}\text{ joules.}$$

29.17. Potential difference between the plates

$$\Delta V = EL = (1.92\times10^6\text{ nt/coul})(0.015\text{ meter})$$

$$= 2880\text{ volts.}$$

29.18. (*a*) The potential at the point *P* on the ring of charge radius *a* can be computed from

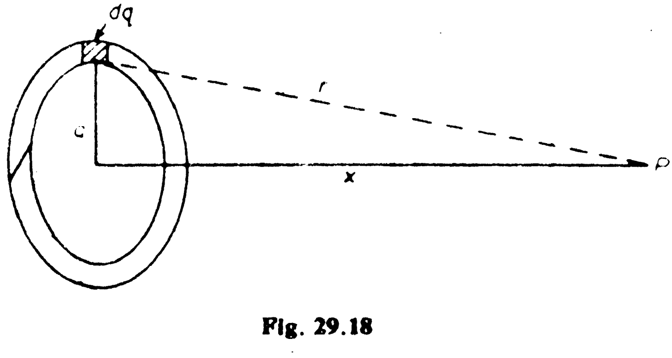


Fig. 29.18

$$V = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0 r}\int dq = \frac{q}{4\pi\varepsilon_0 r}$$

where r is the distance of P from the differential element of charge. Since $r^2=a^2+x^2$,

$$V=\frac{q}{4\pi\varepsilon_0\sqrt{x^2+a^2}}$$

Where x is the distance of P from the center of the ring along the axis.

(*b*) The field is given by

$$E=-\frac{\partial V}{\partial x}=-\frac{\partial}{\partial x}\frac{q}{4\pi\varepsilon_0\sqrt{x^2+a^2}}$$

$$=-\frac{q}{4\pi\varepsilon_0}(-\tfrac{1}{2})(2x)(x^2+a^2)^{-3/2}$$

$$=\frac{qx}{4\pi\varepsilon_0(x^2+a^2)^{3/2}}$$

an expression which is in agreement with that obtained by direct calculation of E in Example 5, Chapter 27.

29.19. $V\frac{\sigma}{2\varepsilon_0}\left(\sqrt{a^2+r^2}-r\right)$

$$E=-\frac{\partial V}{\partial r}=-\frac{\partial}{\partial r}\frac{\sigma}{2\varepsilon_0}\left(\sqrt{a^2+r^2}-r\right)$$

$$=-\frac{\sigma}{2\varepsilon_0}\frac{\partial}{\partial r}\left(\sqrt{a^2+r^2}-r\right)$$

$$=-\frac{\sigma}{2\varepsilon_0}\left[(2r)(\tfrac{1}{2})(a^2+r^2)^{-\frac{1}{2}}-1\right]$$

$$E=-\frac{\sigma}{2\varepsilon_0}\left(1-\frac{r}{\sqrt{a^2+r^2}}\right) \qquad \ldots(1)$$

(*a*) If $r \gg a$, expression (1) may be re-written as

$$E=\frac{\sigma}{2\varepsilon_0}\left(1-\frac{1}{\sqrt{1+\frac{a^2}{r^2}}}\right)$$

$$=\frac{\sigma}{2\varepsilon_0}\left[1-(1-\frac{a^2}{2r^2}+\ldots\ldots)\right]$$

$$=\frac{\sigma a^2}{4\varepsilon_0 r^2}$$

where we have expanded the radical by binomial theorem. Now, the total charge q is given by

$$q=\sigma(\pi a^2)$$

$$\therefore \qquad E=\frac{q}{4\pi\varepsilon_0 r^2}$$

This is the expected result for the field of a point charge. This is reasonable since at longer distances the disk appears as a point.

(*b*) If $r=0$, expression (1) reduces to

$$E=\frac{\sigma}{2\varepsilon_o}.$$

This expression is identical with the field of a charged sheet of infinite extension. Very near the disk, the conditions of an extensive sheet are fulfilled.

29.20. (*a*) For a dipole

$$V=\frac{1}{4\pi\varepsilon_o}\,\frac{p\cos\theta}{r^2}$$

where p is the dipole moment.

$$E_r=-\frac{\partial V}{\partial r}=-\frac{\partial}{\partial r}\left(\frac{1}{4\pi\varepsilon_o}\,\frac{p\cos\theta}{r^2}\right)$$

$$=-\frac{1}{4\pi\varepsilon_o}\,p\cos\theta\,\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)$$

$$=-\frac{1}{4\pi\varepsilon_o}\,p\cos\theta\left(-\frac{2}{r^3}\right)$$

$$=\frac{p\cos\theta}{2\pi\varepsilon_o r^3}$$

(*b*) E_r is zero for $\theta=90°$ or $270°$.

29.21. Field on the surface of the sphere is

$$E=\frac{q}{4\pi\varepsilon_o r^2}=\frac{(4\times10^{-6}\text{ coul})(9\times10^9\text{nt-m}^2/\text{coul}^2)}{(0.1\text{ meter})^2}$$

$$=3.6\times10^6\text{nt/coul}$$

This value exceeds the dielectric strength of 3×10^6 volts/meter or 3×10^6 nt/coul.

29.22. The field E at a distance y from an infinite line of charge density λ is given by

$$E=\frac{\lambda}{2\pi\varepsilon_o y} \qquad \ldots(1)$$

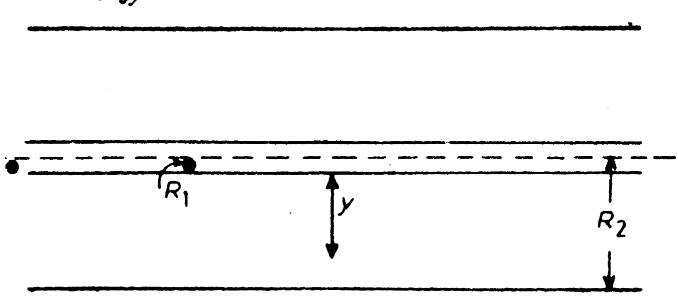


Fig. 29.22

The potential difference between points 1 and 2 is given by

$$\Delta V=V_1-V_2=-\int_{R_2}^{R_1} E dy=-\int_{R_2}^{R_1} \frac{\lambda dy}{2\pi\varepsilon_o y}$$

$$=-\frac{\lambda}{2\pi\varepsilon_o}\int_{R_2}^{R_1} \frac{dy}{y}=-\frac{\lambda}{2\pi\varepsilon_o}\ln\frac{R_1}{R_2}=\frac{\lambda}{2\pi\varepsilon_o}\ln\frac{R_2}{R_1}$$

where R_1 is the radius of the wire and R_2 the radius of the cylinder.

Thus, $$\lambda=\frac{2\pi\varepsilon_o\Delta V}{\ln(R_2/R_1)} \qquad \ldots(2)$$

Use (2) in (1) to get

$$E=\frac{\Delta V}{y\ln(R_2/R_1)}$$

Set $V=850$ volts

$R_1=0.0025$ in $=0.00635$ cm

$R_2=1.0$ cm

Then, $$E=\frac{850 \text{ volts}}{y\ln\left(\frac{1.0}{0.00635}\right)}=\frac{850 \text{ volts}}{y\ln 157.5}=\frac{168}{y} \text{ volt/meter}$$

(*a*) Electric field strength at the surface of the wire is

$$E(R_1)=\frac{168}{R_1}=\frac{168 \text{ volts}}{6.35\times 10^{-5}\text{ meter}}=2.6\times 10^6 \text{ volts/meter}$$

(*b*) Electric field strength at the surface of the cylinder is

$$E(R_2)=\frac{168}{R_2}=\frac{168 \text{ volts}}{1.0\times 10^{-2}\text{ meter}}=1.7\times 10^4 \text{ volts/meter}$$

29.23. (*a*) The charge is assumed to be located at the respective centers of the sphere. The point midway between the centers of the sphere is 1.0 meter from either center. The potental at the mid point is

$$V=\frac{q_1}{4\pi\varepsilon_o r_1}+\frac{q_2}{4\pi\varepsilon_o r_2}$$

But $r_1=r_2=r_0$

$\therefore$ $$V=\frac{1}{4\pi\varepsilon_o r_0}(q_1+q_2)$$

$$=\frac{(9\times 10^9 \text{nt-m}^2/\text{coul}^2)(1\times 10^{-8}\text{coul}-3\times 10^{-8}\text{coul})}{(1.0 \text{ meter})}$$

$$=-180 \text{ volts}.$$

(*b*) The potential at the surface of sphere 1 is due to its own charge q_1 plus that of q_2 of sphere 2 acting at distance r.

$$V=\frac{q_1}{4\pi\varepsilon_o R}+\frac{q_2}{4\pi\varepsilon_o r}$$

$$=(9\times 10^9 \text{ nt-m}^2/\text{coul}^2)\left(\frac{1.0\times 10^{-8}\text{coul}}{0.03 \text{ meter}}-\frac{3\times 10^{-8}\text{coul}}{2.0 \text{ meters}}\right)$$

$$=2865 \text{ volts}$$

The potential at the surface of sphere 2 is due to its own charge q_2 plus that of q_1 of sphere 1 acting at distance r.

$$V=\frac{q_2}{4\pi\varepsilon_o R}+\frac{q_1}{4\pi\varepsilon_o r}$$

$$=(9\times 10^9 \text{nt-m}^2/\text{coul}^2)\left(\frac{-3\times 10^{-8}\text{coul}}{0.03 \text{ meter}}+\frac{1\times 10^{-8}\text{coul}}{2.0 \text{ meter}}\right)$$

$$=-8955 \text{ volts}.$$

29.24. Let the total charge $q=q_1+q_2$ ...(1)

After the spheres are connected the potentials of the two spheres are equal.

$$\frac{1}{4\pi\varepsilon_o}\frac{q_1}{R_1}=\frac{1}{4\pi\varepsilon_o}\frac{q_2}{R_2} \quad ...(2)$$

Also, $$\frac{q_1}{q_2}=\frac{R_1}{R_2} \quad ...(3)$$

where q_1 is the charge on the sphere of radius R_1 and q_2 is the charge on the sphere of radius R_2.

The surface charge densities for the spheres are given by

$$\sigma_1=\frac{q_1}{4\pi R_1^2} \text{ and } \sigma_2=\frac{q_2}{4\pi R_2^2} \quad ...(4)$$

By Problem, $R_1=1.0$ cm ; $R_2=2.0$ cm and $q=2.0\times 10^{-7}$ coul.

(*a*) Eliminate q_2 between (1) and (3),

$$\frac{q_1}{q-q_1}=\frac{R_1}{R_2}$$

$$\frac{q_1}{(2\times 10^{-7})-q_1}=\frac{1.0 \text{ cm}}{2.0 \text{ cm}}=\frac{1}{2}$$

whence $q_1=6.7\times 10^{-8}$ coul (small sphere).

Also, $q_2=1.3\times 10^{-7}$ coul (large sphere).

(*b*) Using (4),

$$\sigma_1=\frac{6.7\times10^{-8}\ \text{coul}}{4\pi(0.01\ \text{meter})^2}=5.3\times10^{-5}\ \text{coul/m}^2\ \text{(small sphere)}$$

$$\sigma_2=\frac{1.3\times10^{-7}\ \text{coul}}{4\pi(0.02\ \text{meter})^2}=2.6\times10^{-5}\ \text{coul/m}^2\ \text{(large sphere)}$$

(*c*) Using (2),

$$V=(9\times10^9\text{nt-m}^2/\text{coul}^2)\left(\frac{6.7\times10^{-8}\ \text{coul}}{0.01\ \text{meter}}\right)$$

$$=6\times10^4\ \text{volts}$$

$$=60\ \text{kv.}$$

29.25. There will be a greater density of charge in regions of large curvature and a lower charge density on surfaces of small curvature. Since the electric field intensity near a point charge is proportional to the charge, electric field will be largest near points where the charge density is greatest. Accordingly, lines of force may be drawn by spacing them more closely in places the charge density is larger. As the surface of the conductor is an equipotential surface, the lines of force are normal to the surface. These are shown solid lines pointing inward in Fig. 29.25. The intersection of the equipotential surfaces, with the plane of figure shown as dashed lines are every where normal to the lines of force.

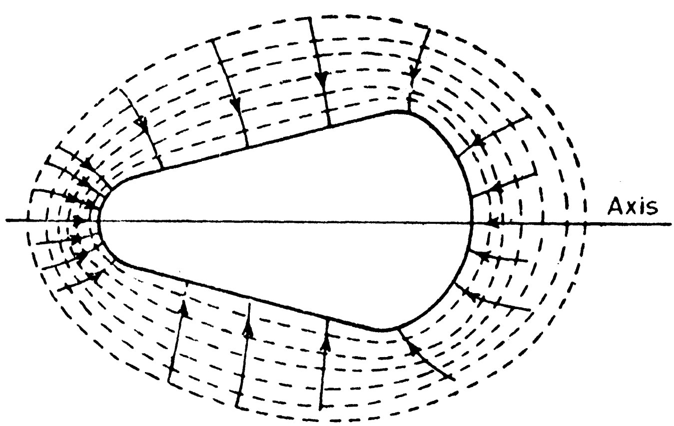


Fig. 29.25

29.26. Let the charges $+q, -q$ and $+q$ be at the vertices of an isosceles triangle of sides $2a$, $2a$ and a (Fig. 29.26). The potential energy of this configuration is

$$U=\frac{1}{4\pi\varepsilon_0}\left[\frac{q^2}{a}-\frac{q^2}{2a}-\frac{q^2}{2a}\right]=0$$

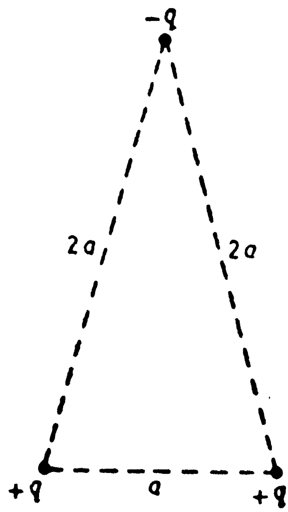


Fig. 29.26

29.27. The potential energy of the configuration taking charges in pairs, is

$$U=\frac{1}{4\pi\varepsilon_0}\left(-\frac{q^2}{a}+\frac{q^2}{\sqrt{2}a}-\frac{q^2}{a}-\frac{q^2}{a}+\frac{q^2}{\sqrt{2}a}-\frac{q^2}{a}\right)$$

$$=-\frac{0.21q^2}{\varepsilon_0\, a}$$

where potential energy at ∞ is taken as zero. This is the work required to put the four charges together.

29.28. When the α-particle just touches the surface of gold nucleus, the original kinetic energy is completely transformed into electric potential energy.

$$U=\frac{q_1\, q_2}{4\pi\varepsilon_0\, r}$$

Set $q_1=2e$;

$q_2=79e$, where e is the proton charge, and

$r=5\times10^{-15}$ meter

$$U=\frac{(2\times1.6\times10^{-19}\text{coul})(79\times1.6\times10^{-19}\text{coul})(9\times10^{9}\text{nt-m}^2/\text{coul}^2)}{(5\times10^{-15}\text{ meter})}$$

$=7.3\times10^{-12}$ joules

$=(7.3\times10^{-12}\text{ joules})/\left(1.6\times10^{-13}\ \dfrac{\text{joule}}{\text{Mev}}\right)$

$=45.6$ Mev

The α-particles used in the experiments of Rutherford stayec well outside the radius of gold nucleus. Rutherford could only put an upper limit for the estimation of nuclear radius of gold nucleus since "anamolous scattering" did not show up with α-particles of 5 Mev energy.

29.29. The potential gradient at distance r is given by

$$\frac{q}{4\pi\varepsilon_o\, r^2}=\frac{(79\times1.6\times10^{-19}\text{ coul})(9\times10^9\text{ nt-m}^2/\text{coul}^2)}{(10^{-12}\text{ meter})^2}$$

$$=1.14\times10^{17}\text{ volts/meter}$$

The potential grdaient at the surface of gold nucleus of radius $R=5\times10^{-15}$ meter is

$$\frac{q}{4\pi\varepsilon_o\, R^2}=\frac{(79\times1.6\times10^{-19}\text{ coul})(9\times19^9\text{ nt-m}^2/\text{coul}^2)}{(5\times10^{-15}\text{ meter})^2}$$

$$=4.55\times10^{21}\text{ volts/meter}$$

29.30. $E_1=\dfrac{q_1}{4\pi\varepsilon_o\, R_1^2}$

$$E_2=\frac{q_2}{4\pi\varepsilon_o\, R_2^2}$$

$$\therefore\quad \frac{E_1}{E_2}=\frac{q_1\, R_2^2}{q_2\, R_1^2}$$

But $\dfrac{q_1}{q_2}=\dfrac{R_1}{R_2}$

$$\therefore\quad \frac{E_1}{E_2}=\frac{R_2}{R_1}$$

29.31. (*a*) Energy acquired by an electron in falling through a potential difference of V volts is

$$eV=\tfrac{1}{2}mv^2$$

Set $\quad v=c$

Then, $\quad V=\dfrac{1}{2}\dfrac{m}{e}c^2=\dfrac{1}{2}\dfrac{(9.1\times10^{-31}\text{kg})(3\times10^8\text{ meter/sec})^2}{(1.6\times10^{-19}\text{ coul})}$

$$=2.6\times10^5\text{ volts}$$

(*b*) $$eV=K=mc^2\left[\frac{1}{\sqrt{1-(v/c)^2}}-1\right]$$

$$\therefore\quad \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}=1+\frac{eV}{mc^2}=1+\frac{1}{2}=\frac{3}{2}$$

whence $\quad \dfrac{v}{c}=\dfrac{\sqrt{5}}{4}=0.745$

$$v=0.745\,c=(0.745)(3\times 10^{8}\ \text{meter/sec})$$

$$=2.236\times 10^{8}\ \text{meter/sec.}$$

29.32. (*i*) $r<R_1$

As no charge is enclosed inside the sphere of radius R_1, we conclude that $E=0$.

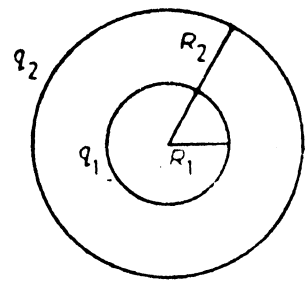


Fig. 29.32 (a)

Due to smaller sphere, $V_1=\dfrac{q_1}{4\pi\varepsilon_o\, R_1}$

and due to larger sphere, $V_2=\dfrac{q_2}{4\pi\varepsilon_o\, R_1}$

$$\therefore \qquad V=V_1+V_2=\frac{1}{4\pi\varepsilon_o}\left(\frac{q_1}{R_1}+\frac{q_2}{R_2}\right)$$

(*ii*) $R_1<r<R_2$

Due to smaller sphere, $V_1=\dfrac{q_1}{4\pi\varepsilon_o r}$

and due to larger sphere, $V_2=\dfrac{q_2}{4\pi\varepsilon_o\, R_2}$

$$\therefore \qquad V=V_1+V_2=\frac{1}{4\pi\varepsilon_o}\left(\frac{q_1}{r}+\frac{q_2}{R_2}\right)$$

$$E=-\frac{\partial V}{\partial r}=\frac{q_1}{4\pi\varepsilon_o r^2}$$

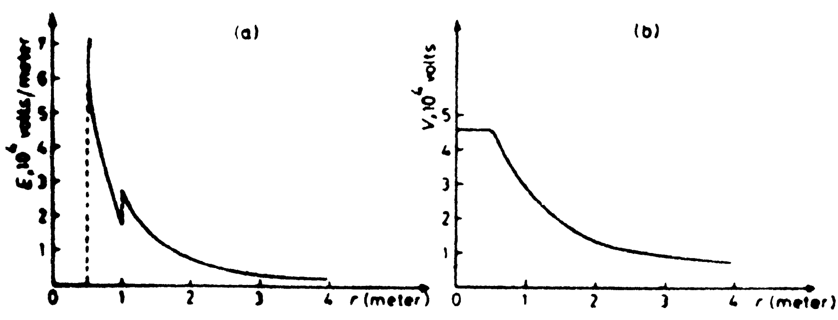


Fig 29.32 (b)

(*iii*) $r>R_2$

Due to smaller sphere, $V_1=\dfrac{q_1}{4\pi\varepsilon_o r}$

and due to larger sphere, $V_2 = \frac{q_2}{4\pi\varepsilon_0 r}$

$$\therefore \qquad V = V_1 + V_2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1+q_2}{r}\right)$$

$$E = -\frac{\partial V}{\partial r} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1+q_2}{r^2}\right)$$

Figure 29.32(b) shows that plots of $E(r)$ and $V(r)$ from $r=0$ to $r=4.0$ meters for $R_1=0.5$ meters, $R_2=1.0$ meter, $q_1=+2.0\times10^{-6}$ coul and $q_2=+1.0\times10^{-6}$ coul.

29.33. Power delivered to the belt,

$$\begin{aligned} P &= \text{rate of energy transfer} \\ &= (q/\text{unit time})(V) \\ &= (3\times10^{-3}\ \text{coul/sec})(3\times10^{6}\ \text{volts}) \\ &= 9000\ \text{watts} \\ &= 9.0\ \text{kilowatts.} \end{aligned}$$

29.34. (*a*) $V = \frac{q}{4\pi\varepsilon_0 r}$

Set, $\quad q = 4\pi\varepsilon_0 r V$

$r=1.0$ meter and $V=1.0\times10^{-6}$ volts,

$$q = \frac{(1.0\ \text{meter})(1.0\times10^{6}\ \text{volts})}{(9\times10^{9}\ \text{nt-m}^2/\text{coul}^2)} = 1.1\times10^{-4}\ \text{coul}$$

For $r=1.0$ cm $=1.0\times10^{-2}$ meter,

$$q = \frac{(1.0\times10^{-2}\ \text{meter})(1.0\times10^{6}\ \text{volts})}{9\times10^{9}\ \text{nt}-\text{m}^2/\text{coul}^2)} = 1.1\times10^{-6}\ \text{coul}$$

(*b*) Owing to a larger value of E on the surface of a smaller sphere, charge would leak out rapidly.

29.35. (*a*) Kinetic energy acquired by the α-particle is

$$\begin{aligned} K_\alpha &= qV = 2eV = (2)(1.6\times10^{-19}\ \text{coul})(1.0\times10^{6}\ \text{volts}) \\ &= 3.2\times10^{-13}\ \text{joules} \end{aligned}$$

(*b*) Kinetic energy acquired by the proton is

$$\begin{aligned} K\alpha &= qV = eV = (1.6\times10^{-19}\ \text{coul})(1.0\times10^{6}\ \text{volt}) \\ &= 1.6\times10^{-13}\ \text{joules} \end{aligned}$$

(*c*) From (*a*) and (*b*) we find

$$\frac{K_p}{K_\alpha} = \frac{\frac{1}{2}M_p v_p^2}{\frac{1}{2}M_\alpha v_\alpha^2} = \frac{eV}{2eV} = \frac{1}{2}$$

$$\therefore \quad \frac{v_p}{v_\alpha}=\sqrt{\frac{1}{2}\frac{M_\alpha}{M_p}}=\sqrt{\frac{1}{2}\times\frac{4}{1}}=\sqrt{2}$$

That is, $\quad v_p > v_\alpha$

The proton has greater speed than α-particle.

SUPPLEMENTARY PROBLEMS

S.29.1. The charge on the surface of the conducting sphere of radius r is

$$q=4\,\pi\varepsilon_o\, rV$$

where V is the potential.

Surface charge density, $\sigma=\dfrac{q}{4\,\pi r^2}=\dfrac{V\varepsilon_o}{r}$

$$=\frac{(200\text{ volt})\,(8.9\times10^{-12}\text{ coul}^2/\text{nt-m}^2)}{(0.15\text{ meter})}$$

$$=1.2\times10^{-8}\text{ coul/meter}^2$$

S.29.2. As the conducting spheres are far apart (10 meters), we can ignore the influence of one sphere on the other in altering the potential. The potential on individual spheres would be caused by the charge residing on a particular sphere. On the sphere with $V=+1500$ volt,

$$q=4\pi\varepsilon_o\,Vr$$

$$=\frac{(1500\text{ volt})\,(0.15\text{ meter})}{(9\times10^9\text{ nt-m}^2/\text{coul}^2)}=2.5\times10^{-8}\text{ coul.}$$

On the sphere with $V=-1500$ volt,

$$q=-2.5\times10^{-8}\text{ coul.}$$

S.29.3. (*a*) If the spheres are connected by a conducting wire, charge will flow from the smaller sphere (higher potential) to the larger sphere (lower potential) until the spheres acquire the same common potential. Let charge q be transfered from the smaller to the larger sphere. The smaller sphere will now have final charge $q_1=q_o-q$ and the large sphere will have final charge $q_2=q_0+q$, where q_o is the initial charge on either sphere.

Let the common potential be V.

$$V=\frac{q_1}{4\,\pi\varepsilon_o r_1}=\frac{q_2}{4\,\pi\varepsilon_o\,r_2}$$

$$\frac{q_1}{q_2}=\frac{r_1}{r_2}=\frac{6.0\text{ cm}}{12.0\text{ cm}}=\frac{1}{2}$$

$$2\,q_1=q_2$$

$$2\,(q_o-q)=q_o+q$$

whence $q=\frac{q_0}{3}=\frac{3\times 10^{-8}\text{ coul}}{3}=1.0\times 10^{-8}$ coul.

(*b*) The final charge on the smaller sphere is

$$q_1=q_0-q_0=(3\times 10^{-8}-1\times 10^{-8})\text{ coul}=2\times 10^{-8}\text{ coul.}$$

The final charge on the larger sphere is

$$q_2=q_0+q=(3\times 10^{-8}+1\times 10^{-8})\text{ coul}=4\times 10^{-8}\text{ coul}$$

The common potential finally reached is

$$V=\frac{q_1}{4\pi\varepsilon_0 r_1}$$

$$=\frac{(9\times 10^9\text{ nt-m}^2/\text{coul}^2)(2\times 10^{-8}\text{ coul})}{(0.06\text{ meter})}=3000\text{ volts.}$$

S.29.4. Let the sides of the rectangle be $a=5$ cm and $b=15$ cm.

(*a*) The electric potential at corner *B* is

$$V_B=\frac{q_1}{4\pi\varepsilon_0 a}+\frac{q_2}{4\pi\varepsilon_0 b}=\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{a}+\frac{q_2}{b}\right)$$

$$=(9\times 10^9\text{nt-m}^2/\text{coul}^2)\left(-\frac{5\times 10^{-6}\text{ coul}}{5\times 10^{-2}\text{ meter}}+\frac{2\times 10^{-6}\text{ coul}}{15\times 10^{-2}\text{ meter}}\right)$$

$$=-7.8\times 10^5\text{ volt}$$

The electric potential at corner *A* is

$$V_A=\frac{q_1}{4\pi\varepsilon_0 b}+\frac{q_2}{4\pi\varepsilon_0 a}=\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{b}+\frac{q_2}{a}\right)$$

$$=(9\times 10^9\text{ nt-m}^2/\text{coul}^2)\left(-\frac{5\times 10^{-6}\text{ coul}}{15\times 10^{-2}\text{ meter}}+\frac{2\times 10^{-6}\text{ coul}}{5\times 10^{-2}\text{ meter}}\right)$$

$$=6\times 10^4\text{ volts.}$$

(*b*) $\Delta V=V_A-V_B=6\times 10^4\text{ volt}-(-7.8\times 10^5\text{ volt})$

$$=8.4\times 10^5\text{ volt}$$

Work done, $W=q_3\,\Delta V=(3\times 10^{-6}\text{ coul})\,(8.4\times 10^5\text{ volt})$

$$=2.52\text{ joules}$$

(*c*) External work is converted into electrostatic potential energy since positive charge is moving from lower to higher potential.

S.29.5. Let the charges be q_1, q_2, q_3 and q_4 each being equal to q. The distance between any pair of charges is the same being equal to a_0. The potential energy for the given configuration is

$$U=\frac{1}{4\pi\varepsilon_0}\left[\frac{q_1q_2}{a}+\frac{q_1q_3}{a}+\frac{q_1q_4}{a}+\frac{q_2q_3}{a}+\frac{q_2q_4}{a}+\frac{q_3q_4}{a}\right]$$

Set $q_1=q_2=q_3=q_4=q$

Then, $U=\frac{6\,q^2}{4\,\pi\varepsilon_0\,a}$

S.29.6. The potential at A due to the charges at B and C is

$$V_A = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{a}+\frac{q}{a}\right)$$

$$= \frac{2q}{4\pi\varepsilon_0 a}$$

where $a=1$ meter. Let the charges at B and C be fixed and the remaining one be moving from A to D.

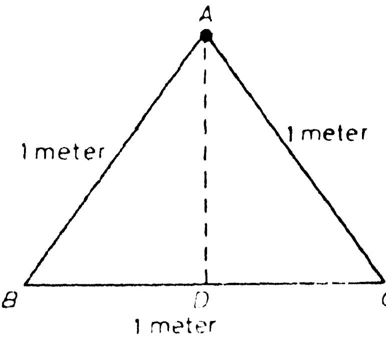


Fig. S.29.6

At the mid-point D of BC,

$$V_D = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\frac{1}{2}a}+\frac{q}{\frac{1}{2}a}\right)$$

$$= \frac{4q}{4\pi\varepsilon_0 a}$$

Potential difference,

$$\Delta V = V_D - V_A = \frac{4q}{4\pi\varepsilon_0 a} - \frac{2q}{4\pi\varepsilon_0 a} = \frac{2q}{4\pi\varepsilon_0 a}$$

$$= \left((9\times10^9\ \frac{\text{nt-m}^2}{\text{coul}^2}\right)\frac{(2)\ (0.1\ \text{coul})}{(1.0\ \text{meter})} = 1.8\times10^9\ \text{volt}.$$

Work done in taking the charge from A to D is

$$W = q\,\Delta V = (0.1\ \text{coul})\ (1.8\times10^9\ \text{volt})$$

$$= 1.8\times10^8\ \text{joules}$$

Energy supplied is 1 kW = 1000 watts = 1000 joules/sec.

∴ Time taken to move the charge from A to D is

$$t = \frac{\text{work done}}{\text{rate of supply of energy}}$$

$$= \frac{1.8\times10^8\ \text{Joules}}{1000\ \text{joules/sec}} = 1.8\times10^5\ \text{sec}$$

$= 50$ hours.

S.29.7. Suppose the density of lines of force increases in the transverse direction i.e. upward (along y-axis) in Fig. S 29.7. Consider a closed path in the form of the rectangle $ABCD$. Let the density of lines along BC be σ_1 and that along AD be σ_2. The electric intensity along BC will then be $E_1 = \frac{\sigma_1}{\varepsilon_0}$ and that along AD will be $E_2 = \frac{\sigma_2}{\varepsilon_0}$.

The potential difference between B and C is $V_{BC} = E_1 d = \frac{\sigma_1 d}{\varepsilon_0}$, where d is the distance BC. Similarly, the potential difference between D and A is $V_{DA} = -E_2 d = -\frac{\sigma_2 d}{\varepsilon_0}$. Let us take a test charge q along

the indicated path. Along AB and CD no work is done as the paths lie on equipotential surfaces. Work done in moving the charge from B to C is

$$W_{BC}=qV_{BC}=\frac{q\,\sigma_1 d}{\varepsilon_0}.$$

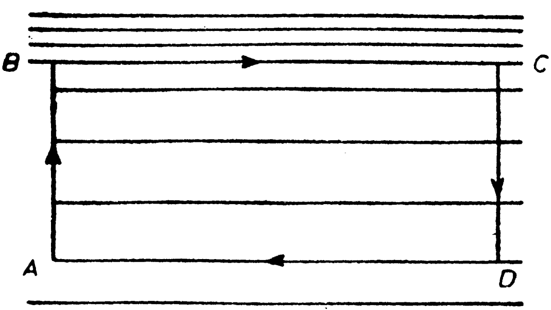


Fig. S.29.7

Similarly, work done in taking the charge from D to A is

$$W_{DA}=qV_{DA}=-\frac{q\sigma_2 d}{\varepsilon_0}$$

Therefore, the work done in the round trip $ABCDA$, is

$$W=\frac{q\sigma_1 d}{\varepsilon_0}-\frac{q\sigma_2 d}{\varepsilon_0}=\frac{qd}{\varepsilon_0}(\sigma_1-\sigma_2)$$

But $\sigma_1 > \sigma_2$, by our postulate. Hence, $W \neq 0$. However, because of the conservative character of the field W should be zero. We, therefore, conclude that our assumption is wrong. Thus, the density of lines at right angles to the lines of force cannot change for an electric field in which all the lines of force are striaght parallel lines.

S.29.8. Electric field near a long line of positive charge is

$$E=\frac{\lambda}{2\,\pi\varepsilon_0\, r} \qquad \ldots(1)$$

The potential at a point P at distance r_1 is given by

$$V=\int E dr=\frac{\lambda}{2\pi\varepsilon_0}\int\frac{dr}{r}=\frac{\lambda}{2\,\pi\varepsilon_0}\ln r_1+C \qquad \ldots(2)$$

where C is a constant of integration.

The potential with the origin O is given by

$$V_o=\frac{\lambda}{2\pi\varepsilon_0}\ln a+C \qquad \ldots(3)$$

The absolute potential V_P at any field point P is given by the potential difference between P due to positive charge and the origin O at zero potential. Subtracting (2) from (3),

$$V_P^+=V_o-V=\frac{\lambda}{2\,\pi\varepsilon_0}\ln\frac{a}{r_1} \qquad \ldots(4)$$

The superscript refers to the positive line of charge. Similarly, for the line of negative charge

$$V_p^- = \frac{(-\lambda)}{2\,\pi\varepsilon_o} \ln \frac{a}{r_2} \qquad ...(5)$$

where r_2 is the distance of P from the negative line of charge. The net potential at the field point P is given by the algebraic sum of the two potentials, V_p^+ and V_p^-. Hence,

$$V_p = V^+{}_p + V^-{}_p = \frac{\lambda}{2\,\pi\varepsilon_o} \ln \frac{r_2}{r_1} \qquad ...(6)$$

Since λ and ε_o are constants, we obtain the equation of an equipotential surface by assigning some value to V_p, either positive or negative.

Re-writing (6), we get,

$$\frac{r_2}{r_1} = e^{\frac{2\,\pi\varepsilon_o V_p}{\lambda}} = C \qquad ...(7)$$

where C is a constant for any fixed value of V_p. Now, the locus of points with a constant ratio of the distances to two lines is an equation to a cylinder. We, therefore, conclude that the equipotential surfaces in this field are a series of cylinders along each line of charge. However, the cylinders are not concentric. Further, an equal negative potential V_p gives rise to a cylinder of the same size but surrounding the negative rather than positive line of charge. Fig. S.29.8 shows some of the equipotential surfaces in the xy plane.

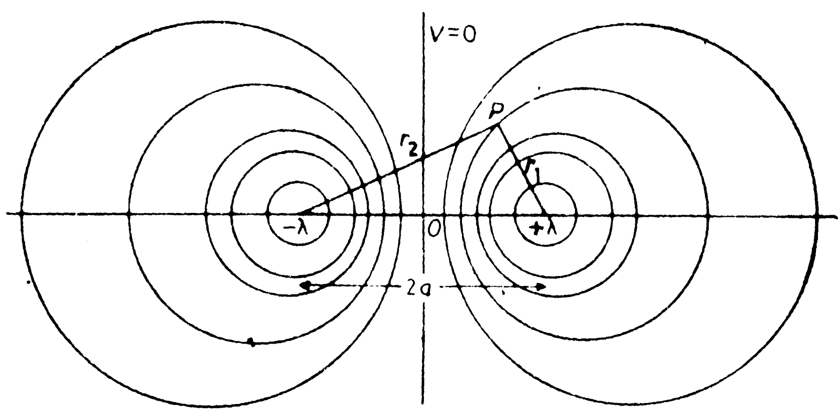


Fig. S.29.8

S.29.9. In the circular orbit of radius r_1, the kinetic energy is $K_1 = \frac{1}{2}\, mv^2{}_1$ and the potential energy $U_1 = -\dfrac{Qq}{4\pi\,\varepsilon_o\, r_1}$. Hence, the total energy is

$$E_1 = K_1 + U_1 = \tfrac{1}{2}\, mv^2{}_1 - \frac{Qq}{4\,\pi\varepsilon_o\, r_1} \qquad ...(1)$$

As the centripetal force is provided by the coulomb force, we have

$$\frac{mv^2_1}{r_1}=\frac{Qq}{4\,\pi\varepsilon_o\,r^2_1}$$

$$mv^2_1=\frac{Qq}{4\,\pi\varepsilon_o\,r_1} \qquad \ldots(2)$$

Using (2) in (1), and simplifying,

$$E_1=\frac{Qq}{8\pi\varepsilon_o\,r_1}-\frac{Qq}{4\pi\varepsilon_o\,r_1}=-\frac{1}{8\pi\varepsilon_o}\frac{Qq}{r_1} \qquad \ldots(3)$$

Similarly, for the circular orbit of radius r_2, we have

$$E_2=-\frac{1}{8\,\pi\varepsilon_o}\frac{Qq}{r_2}$$

The work W that must be done by an external agent on the second particle in order to increase the radius of the circular orbit from r_1 to r_2 is

$$W=E_2-E_1=\frac{Q\,q}{8\pi\varepsilon_o}\left(\frac{1}{r_1}-\frac{1}{r2}\right)$$

S.29.10. Loss of potential energy in moving from r_1 to r_2 is

$$\Delta U=\frac{Q^2}{4\pi\varepsilon_o}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)=\frac{Q^2}{4\pi\varepsilon_o}\left(\frac{r_2-r_1}{r_1\,r_2}\right)$$

Gain in kinetic energy is

$$\Delta K=\tfrac{1}{2}mv^2$$

Gain in kinetic energy = Loss in potential energy

$$\Delta K=\Delta U$$

$$\tfrac{1}{2}mv^2=\frac{Q^2}{4\pi\varepsilon_o}\qquad\frac{(r_2-r_1)}{r_1r_2}$$

$$v=\sqrt{\frac{2Q^2}{4\pi\varepsilon_o}\qquad\frac{(r_2-r_1)}{r_1r_2}}$$

$$=\sqrt{\frac{(2)(9\times10^9\text{ nt-m}^2/\text{coul}^2)(3.1\times10^{-6}\text{ coul})^2(25\times10^{-4}\text{ m}-9\times10^{-4}\text{m})}{(2\times10^{-5}\text{kg})(9\times10^{-4}\text{ meter})(25\times10^{-4}\text{ meter})}}$$

$=2.48\times10^3$ meter/sec

S.29.11. (*a*) As both, the projected particle and the nucleus, are positively charged, the electrical forces are repulsive. If the aim is perfect then the particle will proceed head-on towards the nucleus. As it does so it loses kinetic energy and gains potential energy. Distance of closest approach corresponds to the situation where the particle momentarily comes to stop. In that case the initial

kinetic energy is completely transformed into coulomb potential energy. When the particle has initial kinetic energy, its potential energy is zero as the particle is at infinity. Conservation of energy demands that

$$K=\frac{qQ}{4\pi\varepsilon_0 r_0} \quad \ldots(1)$$

where r_0 is the distance of closest approach.

$$\therefore \quad r_0=\frac{qQ}{4\pi\varepsilon_0 K}$$

(*b*) This is the case of glancing collision. Nowhere does the kinetic energy vanish. Let v be the speed of the particle at P, the distance of closest approach at P from the center of the nucleus being R. Conservation of energy implies that

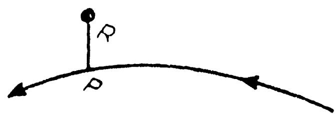


Fig S. 29.11

$$K=\tfrac{1}{2}mv^2+\frac{qQ}{4\pi\varepsilon_0 R} \quad \ldots(2)$$

But by Problem, $R=2r_0$ ...(3)

Combining (1), (2) and (3), we get

$$K=\tfrac{1}{2}mv^2+\tfrac{1}{2}K$$

whence $v=\sqrt{\dfrac{K}{m}}$

S.29.12. Field strength $=\dfrac{\text{potential difference}}{\text{distance moved}}$

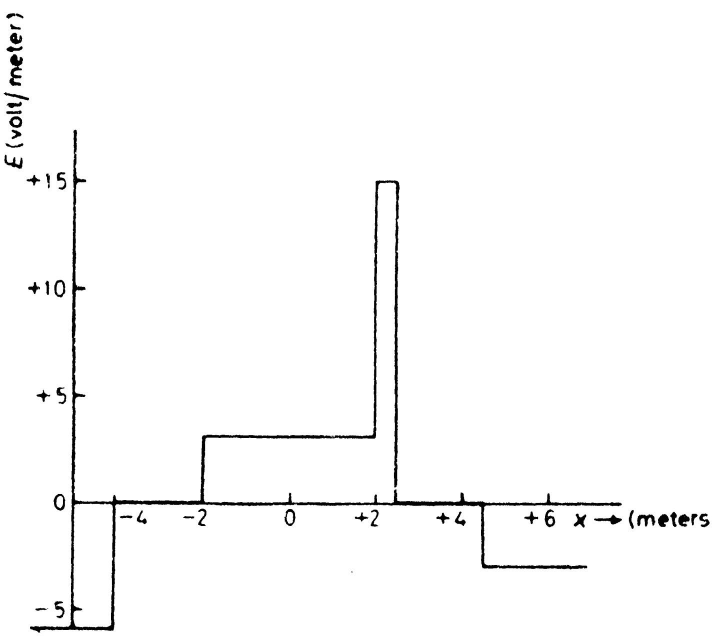


Fig. S.29.12

$$E=-\frac{\Delta V}{\Delta s} \qquad \ldots(1)$$

The minus sign is used in the above definition so that E is positive when $\Delta V/\Delta s$ is negative.

Formula (1) is used to find E cenetered around various intervals of distance. The calculated fields are plotted in Fig. S.29.12 as a histogram.

S.29.13. (*a*) Consider a differential element ds of the segment at distance s from the end closer to P. The charge associated with the element ds is $dq=\lambda\, ds$ ...(1)

The differential potential at P due to the charge element dq is

$$dV=\frac{dq}{4\pi\varepsilon_0\,(y+s)}=\frac{\lambda ds}{4\pi\varepsilon_0\,(y+s)} \qquad \ldots(2)$$

where use has been made of (1).

The potential at P is obtained by integrating (2).

$$V=\int dV=\frac{\lambda}{4\pi\varepsilon_0}\int_0^L\frac{ds}{(y+s)}$$

$$=\frac{\lambda}{4\pi\varepsilon_0}\ln(y+s)\Big|_{s=0}^{s=L}$$

$$=\frac{\lambda}{4\pi\varepsilon_0}\Big[\ln(y+L)-\ln y\Big]$$

$$=\frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{y+L}{y}\right]$$

$$=\frac{\lambda}{4\pi\varepsilon_0}\ln\left(1+\frac{L}{y}\right)$$

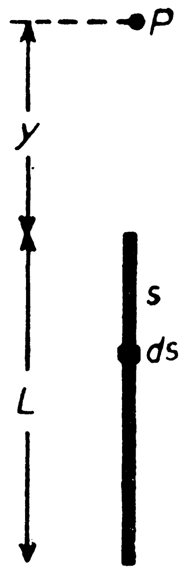


Fig S.29.13

(*b*) $$E_y=-\frac{\partial V}{\partial y}=\frac{-\lambda}{4\pi\varepsilon_0}\,\frac{(-L/y^2)}{1+L/y}=\frac{1}{4\pi\varepsilon_0}\,\frac{\lambda L}{y\,(y+L)}$$

(*c*) $$E_x=-\frac{\partial V}{\partial y}=0$$

S.29.14. (*a*) Consider an element of length dx at distance x from 0, along the length of rod. The charge in dx is

$$dq=\lambda dx=kxdx \qquad \ldots(1)$$

$$dV=\frac{dx}{4\pi\varepsilon_0\sqrt{y^2+x^2}} \qquad \ldots(2)$$

The potential at P is given by

$$V=\int \frac{dq}{4\pi\varepsilon_o \sqrt{y^2+x^2}}=\frac{k}{4\pi\varepsilon_o}\int_0^L \frac{xdx}{\sqrt{y^2+x^2}} \qquad ...(3)$$

Set $\quad z^2=y^2+x^2$ and $zdz=xdx$ in (3)

$$\therefore \quad V=\frac{k}{4\pi\varepsilon_o}\int \frac{zdz}{z}$$

$$=\frac{kz}{4\pi\varepsilon_o}=\frac{k}{4\pi\varepsilon_o}\sqrt{x^2+y^2}=\Bigg|_0^L \frac{k[\sqrt{L^2+y^2}-y]}{4\pi\varepsilon_o} \qquad ...(4)$$

(*b*) $$E_y=-\frac{\partial V}{\partial y}=-\frac{k}{4\pi\varepsilon_o}\left[\frac{y}{\sqrt{L^2+y^2}}-1\right]$$

Aliter

$$dE_x=dE\cos\theta \qquad ...(5)$$

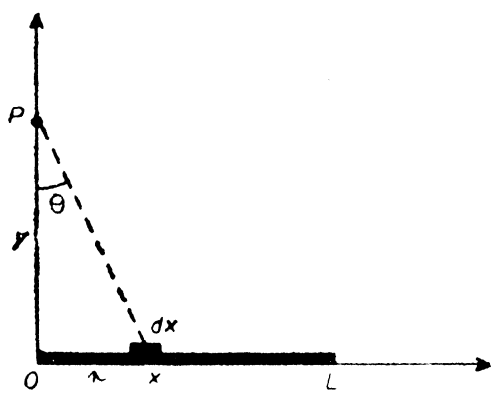


Fig S.29.14

But $\quad dE=\frac{dq}{4\pi\varepsilon_o (y^2+x^2)} \qquad ...(6)$

and $\quad \cos\theta=\frac{y}{\sqrt{y^2+x^2}} \qquad ...(7)$

Using (1), (6) and (7) in (5)

$$dE_y=\frac{kyxdx}{4\pi\varepsilon_o (y^2+x^2)^{3/2}}$$

Integrating,

$$E_y=\int dE_y=\frac{ky}{4\pi\varepsilon_o}\int_0^L \frac{xdx}{(y^2+x^2)^{3/2}}$$

Set $y^2+x^2=z^2$

$$xdx=zdz$$

$$E_y=\frac{ky}{4\pi\varepsilon_o}\int\frac{zdz}{z_3}=-\frac{ky}{4\pi\varepsilon_o}\quad\frac{1}{z}\Big|$$

$$=-\frac{ky}{4\pi\varepsilon_o}\frac{1}{\sqrt{y^2+x^2}}\Bigg|_0^L=-\frac{ky}{4\pi\varepsilon_o}\left[\frac{y}{\sqrt{L^2+y^2}}-1\right]$$

(*c*) V involves y alone.

30 CAPACITORS AND DIELECTRICS

30.1. (*a*) The equivalent capacitance of two capacitors in series is given by

$$C=\frac{C_1C_2}{C_1+C_2}=\frac{(2\times10^{-6}f)(8\times10^{-6}\text{ f})}{(2\times10^{-6}f+8\times10^{-6}\text{ f})}=1.6\times10^{-6}f$$

The magnitude q of the charge on each plate must be the same.

$q=CV=(1.6\times10^{-6}\text{ f})(300\text{V})$
$=4.8\times10^{-4}\text{coul.}$

The potential difference across 2 μf capacitor is

$V_1=q/C_1=(4.8\times10^{-4}\text{coul})/(2\times10^{-6}\text{ f})$
$=240$ volt

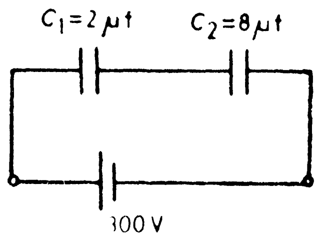


Fig. 30.1

The potential difference across 8 μf capacitor is

$$V_2=q/C_2=(4.8\times10^{-4}\text{ coul}/(8\times10^{-6}\text{ f})=60\text{ volt}$$

(*b*) Total charge, $Q=q+q=2q=2\times4.8\times10^{-4}$ coul
$=9.6\times10^{-4}$ coul

The equivalent capacitance of the capacitors in parallel is given by

$$C=C_1+C_2=2\mu f+8\mu f=10\ \mu f=10^{-5}\text{ f}$$

$\therefore$ The potential difference for each is

$$V=Q/C=9.6\times10^{-4}\text{ coul}/10^{-5}\text{ f}=96\text{ volts.}$$

The charge on 2 μf capacitor is

$$q_1=C_1V=(2\times10^{-6}\text{ f})(96\ V)=1.9\times10^{-4}\text{ coul}$$

The charge on 8 μf capacitor is

$$q_2=C_2V=(8\times10^{-6}\text{ f})(96\ V)=7.7\times10^{-4}\text{ coul}$$

(*c*) The charge is neutralized.

$$q_1=q_2=0$$

Also, $V_1=V_2=0$

30.2. The potential is given by

$$V=\frac{Q}{4\pi\epsilon_0R}$$

$\therefore$ Capacity of earth is

$$C=\frac{Q}{V}=4\pi\varepsilon_o R=\frac{6.4\times10^6 \text{ meter}}{9\times10^9 \text{ nt-m}^2/\text{coul}^2}$$

$$=711\times10^{-6}\text{f}=711\ \mu\text{f}$$

30.3. C_1 is charged and then connected to C_2 by closing switch S. The measured potential difference drops from V_o to V.

$$V=\frac{V_o C_1}{C_1+C_2}$$

whence, $$C_2=\frac{C_1(V_o-V)}{V}=(100\ \mu\mu\text{f})\ \frac{(50 \text{ volt}-35 \text{ volt})}{35 \text{ volt}}$$

$$=43\ \mu\mu\text{f}$$

30.4. Textbook Eq. 30 7 is $C=\frac{q}{V}=\frac{\varepsilon_o A}{d}$

$$\therefore \qquad [\varepsilon_o]=[C]\frac{[L]}{[L^2]}=\frac{[C]}{[L]}$$

$\therefore$ MKS units of ε_o are farad/meter.

But, $$\frac{\text{Farad}}{\text{Meter}}=\frac{\text{coul}}{(\text{volt})(\text{meter})}=\frac{\text{coul}}{(\text{joule/coul})(\text{meter})}$$

$$=\frac{\text{coul}^2}{\text{joul-meter}}=\frac{\text{coul}^2}{\text{nt-meter}^2}$$

30.5. The capacity of upper capacitor is given by

$$C_1=\frac{\varepsilon_o A}{d_1} \qquad \ldots(1)$$

and that of lower one is given by

$$C_2=\frac{\varepsilon_o A}{d_2} \qquad \ldots(2)$$

For the series combi nation of capacitors,the equivalent capacitance is given by

$$C=\frac{C_1C_2}{C_1+C_2} \qquad \ldots(3)$$

Using (1) and (2) in (3) we find,

$$C=\frac{\varepsilon_o A}{d_1+d_2}$$

But, $d_1+d_2=a-b$

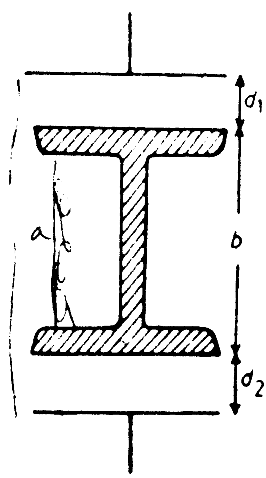


Fig. 30.5

$$\therefore \qquad C=\frac{\varepsilon_0 A}{a-b}.$$

30.6. Since the effective area of the capacitor is that of interleaved portion of the plates only, the maximum effective area of each plate will be A. The neighbouring plates constitute a parallel-plate capacitor of capacitance, $C_0=\frac{\varepsilon_0 A}{d}$; and as $(n-1)$ plates in parallel make up the variable capacitor the capacitor has the maximum capacitance, $C=\frac{(n-1)\,\varepsilon_0 A}{d}$.

30.7. The outer spherical plate is invariably grounded and contact is made with the inner plate through a small hole in the outer one (Fig. 30.7).

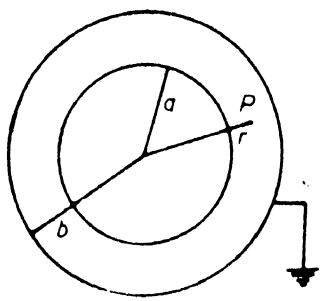


Fig. 30.7

The field at point P is caused entirely by the charge Q on the inner sphere and has the value

$$E=\frac{Q}{4\pi\varepsilon_0 r^2}$$

The potential difference between the two spheres is given by

$$V=-\int_b^a E.dr=-\int_b^a \frac{Q\,dr}{4\pi\varepsilon_0 r^2}=\frac{Q}{4\pi\varepsilon_0 r_2}\int_b^a$$

$$=\frac{Q\,(b-a)}{4\pi\varepsilon_0 ab}$$

whence, $\quad C=\frac{Q}{V}=\frac{4\pi\varepsilon_0 ab}{b-a}$

30.8. Imagine that the capacitor is divided into differential strips which are practically parallel. Consider a strip at distance x (Fig. 30.8) of length a perpendicular to the plane of paper and of width dx in the plane of paper, the area of the strip being $dA=adx$. At the distance x, the separation of the plates is seen to be

$$D=d+x\theta$$

The capacitance due to the differential strip facing each plate is

$$dC=\frac{\varepsilon_0 A}{D}=\frac{\varepsilon_0 a dx}{d+x\theta}$$

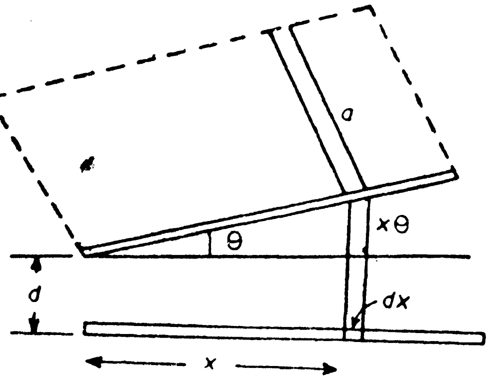


Fig. 30.8

The capacitance is given by

$$C=\int dc=\int_0^a \frac{\varepsilon_0 a\,dx}{(d+x\theta)}=\varepsilon_0 a\int_0^a \frac{dx}{(d+x\theta)}$$

$$=\frac{\varepsilon_0 a}{d}\int_0^a\left(1+\frac{x\theta}{d}\right)^{-1}dx$$

$$=\frac{\varepsilon_0 a}{d}\int_0^a\left(1-\frac{x\theta}{d}+\cdots\right)dx$$

$$=\frac{\varepsilon_0 a}{d}\left(x-\frac{x^2\theta}{2d}\right)\Bigg|_0^a$$

$$=\frac{\varepsilon_0 a^2}{d}\left(1-\frac{a\theta}{2d}\right)$$

30.9. (*a*) For the two concentric spherical shells of radii a and b, the capacitance is

$$C=4\pi\varepsilon_0\frac{ab}{b-a}$$

If $b \simeq a$, than $ab \simeq a^2$ and $b-a=d$

$$\therefore\ C=4\pi\varepsilon_0\frac{a^2}{d}=\frac{\varepsilon_0}{d}4\pi a^2=\frac{\varepsilon_0 A}{d}$$

where we have set $A=4\pi a^2$, the surface area of the sphere.

30.10. Capacitance of the parallel-plate capacitor is

$$C=\frac{\varepsilon_0 A}{d}=\frac{\varepsilon_0\,(\pi r^2)}{d}$$

Charge, $Q=CV=\dfrac{\varepsilon_0\,(\pi r^2)V}{d}$

$$=\frac{(8.9\times 10^{-12}\text{ coul}^2/\text{nt-m}^2)\,(0.08\text{ meter})^2\,(100\text{ volt})\pi}{(1.0\times 10^{-3}\text{ meter})}$$

$$=1.8\times 10^{-8}\text{ coul}$$

30.11. As C_1 and C_2 are in parallel, their equivalent is C_1+C_2. Now C_3 is in series with C_1+C_2. Hence, the equivalent capacitance of the combination is given by

$$\frac{1}{C}=\frac{1}{C_1+C_2}+\frac{1}{C_3}$$

$$C=\frac{C_3\,(C_1+C_2)}{C_1+C_2+C_3}$$

$$C=\frac{(4\ \mu\text{f})\,(10\ \mu\text{f}+5\ \mu\text{f})}{(10\ \mu\text{f}+5\ \mu\text{f}+4\ \mu\text{f})}=3.16\ \mu\text{f}$$

30.12. (*b*) The charge on C_3 is equal to that across the combination of C_1 and C_2 in parallel.

$\therefore\quad C_3V_3=(C_1+C_2)V_1$

where V_3 and V_1 are the corresponding potential difference.

$$V_3=\frac{(C_1+C_2)\,V_1}{C_3}=\frac{(10\ \mu\text{f}+5\ \mu\text{f})V_1}{4\ \mu\text{f}}=\frac{15}{4}V_1 \qquad \cdot(1)$$

Also, $V_1+V_3=100$. $\qquad \ldots(2)$

Solving (1) and (2),

$V_1=21$ volts.

When the capacitor C_3 breaks down, the voltage across C_1 becomes 100 volts. Hence, change in potential difference is

$$\Delta V=(100-21)=79\text{ volts.}$$

(*a*) Change in charge is

$$\Delta Q_1=C_1\Delta V$$

$$=(10\times 10^{-6}\text{ f})\,(79\text{ volts})$$

$$=7.9\times 10^{-4}\text{ coul}$$

30.13. Let the effective capacitance between points x and y be C. Apply a potential difference V between x and y and let the effective capacitance be charged to q. Let the charge across C_1 and C_2 be q_1

and q_2 respectively. The charges across various capacitors are shown in Fig. 30.13.

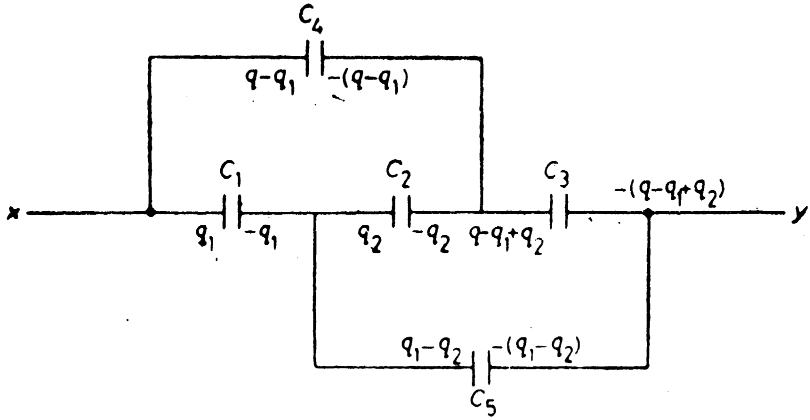


Fig. 30.13

The potential drop across C_1 plus that across C_5 must be equal to the potential drop across C_4 plus that across C_3.

$$V_1+V_5=V_4+V_3=V$$

$$\therefore \frac{q_1}{C_1}+\frac{q_1-q_2}{C_5}=\frac{q-q_1}{C_4}+\frac{q-q_1+q_2}{C_3}=V \qquad \ldots(1)$$

But, by Problem, $C_1=C_3=C_4=C_5$

Multiply (1) through by C_1 to get

$$2q_1-q_2=C_1V \qquad \ldots(2)$$

$$2q-2q_1+q_2=C_1V$$

Add (2) and (3) to find $\ldots(3)$

$$2q=2C_1V \qquad \text{or } C=\frac{q}{V}=C_1=4.0\ \mu f$$

30.14. (*a*) Five 2.5 μf capacitors in series would provide an equivalent capacitance of 0.4 μf. At the same time each will be able to withstand 200 volts without breakdown, Fig 30.4 (*a*).

(*b*) Three arrays, each consisting of five 2.0 μf capacitors in series give the equivalent capacitance of 1.2 μf. At the same time each will be able to withstand 200 volts without breadown, Fig 30.14 (*b*).

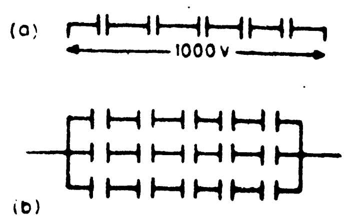


Fig 30.14

30.15. The equivalent capacitance of C_1 and C_2 in series is

$$C_e=\frac{C_1C_2}{C_1+C_2}$$

The equivalent capacitance of C_0 and C_3 in parallel is

$$C=C_0+C_3=\frac{C_1C_2}{C_1+C_2}+C_3$$

$$=\frac{(10\ \mu f)\ (5\ \mu f)}{(10\ \mu f+5\ \mu f)}+4\ \mu f=7.3\ \mu f$$

30.16. (*a*) The equivalent capacitance of C_1 and C_3 in series is

$$C_{13}=\frac{C_1C_3}{C_1+C_3}=\frac{(1\ \mu f)\ (3\ \mu f)}{(1\ \mu f+3\ \mu f)}=0.75\ \mu f$$

Similarly, the equivalent capacitance of C_2 and C_4 in series is

$$C_{24}=\frac{C_2C_4}{C_2+C_4}=\frac{(2\ \mu f)\ (4\ \mu f)}{(2\ \mu f+4\ \mu f)}=1.33\ \mu f$$

The combination of C_{13} and C_{24} in parallel is

$$C=C_{13}+C_{24}=0.75\ \mu f+1.33\ \mu f=2.08\ \mu f$$

Charge on the equivalent capacitor is

$$q=CV=(2.08\times 10^{-6}\ f)\ (12\ \text{volts})=25\times 10^{-6}\ \text{coul.}$$

The charge on C_1 and C_3 will be equal and is

$$q_1=q_3=C_{13}V=(0.75\times 10^{-6}\ f)\ (12\ \text{volt})=9\times 10^{-6}\ \text{coul}$$

The charge on C_2 and C_4 will be equal and is

$$q_2=q_4=C_{24}V=(1.33\times 10^{-6}\ f)\ (12\ \text{volt})=16\times 10^{-6}\ \text{coul.}$$

(*b*) The equivalent of C_1 and C_2 in parallel is

$$C_{12}=C_1+C_2=1\ \mu f+2\ \mu f=3\ \mu f$$

The equivalent of C_3 and C_4 in parallel is

$$C_{34}=C_3+C_4=3\ \mu f+4\ \mu f=7\ \mu f.$$

As C_{12} and C_{34} are in series, the equivalent capacitor is given by

$$C=\frac{C_{12}C_{34}}{C_{12}+C_{34}}=\frac{(3\ \mu f)\ (7\ \mu f)}{(3\ \mu f+7\ \mu f)}=2.1\ f\mu.$$

The charge on the equivalent capacitor is

$$Q=CV=(2.1\times 10^{-6}\ f)\ (12\ \text{volt})=25.2\times 810^{-6}\ \text{coul.}$$

The potential difference across C_1 or C_2 is

$$V_1=V_2=\frac{Q}{C_{12}}=\frac{25.2\times 10^{-6}\ \text{coul}}{3\times 10^{-6}\ f}=8.4\ \text{volt.}$$

The potential difference across C_3 or C_4 is

$$V_3=V_4=\frac{Q}{C_{34}}=\frac{25.2\times 10^{-6}\ \text{coul}}{7\times 10^{-6}\ f}=3.6\ \text{volt.}$$

$\therefore$ The charges are

$$Q_1=C_1V_1=(1\times 10^{-6}\ f)\ (8.4\ \text{volt})=8.4\times 10^{-6}\ \text{coul.}$$

$$Q_2 = C_2V_2 = (2\times10^{-6}\text{ f})\ (8.4\text{ volt}) = 16.8\times10^{-6}\text{ coul}$$
$$Q_3 = C_3V_3 = (3\times10^{-6}\text{ f})\ (3.6\text{ volt}) = 10.8\times10^{-6}\text{ coul}$$
$$Q_4 = C_4V_4 = (4\times10^{-6}\text{ f})\ (3.6\text{ volt}) = 14.4\times10^{-6}\text{ coul}$$

30.17. The given parallel plate capacitor is equivalent to a combination of two parallel plate capacitors each of area $\frac{1}{2}A$ and dielectrics K_1 and K_2. The capacitance is then given by

$$C = C_1 + C_2 = \frac{K_1\,\varepsilon_0\,(\frac{1}{2}A)}{d} + \frac{K_2\,\varepsilon_0\,(\frac{1}{2}A)}{d}$$

$$= \frac{\varepsilon_0 A}{d}\left(\frac{K_1+K_2}{2}\right)$$

Limiting cases:

(*i*) $K_1 = K_2 = K$

$$C = \frac{\varepsilon_0 AK}{d}.$$

(*ii*) $K_1 = K_2 = 1$ (for air)

$$C = \frac{\varepsilon_0 A}{d}.$$

These are the expected results.

30.18. The parallel-plate capacitor may be thought of as two capacitors with dielectrics K_1 and K_2 in series, their capacitance being given by

$$C_1 = \frac{\varepsilon_0 AK_1}{\frac{1}{2}d} \qquad \text{...(1)}$$

$$C_2 = \frac{\varepsilon_0 AK_2}{\frac{1}{2}d} \qquad \text{...(2)}$$

The combined capacitance is then given by

$$C = \frac{C_1C_2}{C_1+C_2} \qquad \text{...(3)}$$

Using (1) and (2) in (3)

$$C = \frac{2\varepsilon_0 A}{d}\left(\frac{K_1K_2}{K_1+K_2}\right)$$

Limiting cases:

(*i*) $K_1 = K_2 = K$

$$C = \frac{\varepsilon_0 AK}{d}$$

(ii) $K_1 = K_2 = 1$ (for air)

$$C = \frac{\varepsilon_o A}{d}$$

These are the expected results.

30.19. The parallel-plate capacitor may be thought of as an arrangement of two capacitators, one consisting of a dielectric K with thickness b and area A, and another with air gap $(d-b)$ and area A. The combined capacitance is then given by

$$C = \frac{C_1 C_2}{C_1 + C_2} \qquad \text{...(1)}$$

With

$$C_1 = \frac{\varepsilon_o AK}{b} \text{ (for the dielectric)} \qquad \text{...(2)}$$

$$C_2 = \frac{\varepsilon_o A}{d-b} \text{ (for air)} \qquad \text{...(3)}$$

Using (2) and (3) in (1)

$$C = \frac{K \varepsilon_o A}{Kd - b(K-1)}$$

Setting $A = 100 \times 10^{-4}$ meter^2 ; $d = 1.0 \times 10^{-2}$ meter

$b = 0.5 \times 10^{-2}$ meter, and $K = 7.0$

$$C = \frac{(7)\,(8.9 \times 10^{-12} \text{ coul}^2/\text{nt-m}^2)\,(100 \times 10^{-4} \text{ meter}^2)}{(7)\,(1.0 \times 10^{-2} \text{ meter}) - (0.5 \times 10^{-2} \text{ meter})(7-1)}$$

$$= 15.6 \times 10^{-12} \text{ f}$$

$$= 15.6\ \mu\mu\text{f}$$

This is in agreement with the rounded off value 16 $\mu\mu$f obtained Example 5 of the textbook.

When $b = 0$, $C = \dfrac{\varepsilon_o A}{d}$,

When $K = 1$, $C = \dfrac{\varepsilon_o A}{d}$

When $b = d$, $C = \dfrac{K \varepsilon_o A}{d}$

These are the expected results.

30.20. Before the slab is introduced the capacitance of the parallel-plate capacitor is given by the usual formula

$$C = \frac{\varepsilon_o A}{d}$$

After the slab of copper is introduced, the original capacitor is reduced to two in series each having a gap of $\frac{1}{2}(d-b)$. Each has capacitance

$$C_1=C_2=\frac{\varepsilon_0 A}{\frac{1}{2}(d-b)}=\frac{2\,\varepsilon_0 A}{d-b}$$

The combination of these two capacitors in series has the capacitance

$$C=\frac{C_1C_2}{C_1+C_2}=\frac{C_1}{2},\text{ with } C_1=C_2.$$

$$\therefore \qquad C=\frac{\varepsilon_0 A}{d-b}$$

30.21. For a parallel-plate capacitor of dielectric K,

$$C=\frac{\varepsilon_0 AK}{d}$$

Since ε_0 and A are constant it is sufficient to find the ratio K/d in order to estimate the relative magnitudes of C.

For mica, $K/d=6/0.1\text{ mm}=60/\text{mm}$

For glass, $K/d=7/2.0\text{ mm}=3.5/\text{mm}$

For paraffin, $K/d=2/10\text{ mm}=0.2/\text{mm}$.

Clearly, to obtain the largest capacitance, we must place the mica sheet.

30.22. (*d*) Before the dielectric slab is introduced, the capacitance

$$C_0=\frac{\varepsilon_0 A}{d}$$

$$=\frac{(8.9\times 10^{-12}\text{ coul}^2/\text{nt-m}^2)(10^{-2}\text{ meter}^2)}{(10^{-2}\text{ meter})}=8.9\times 10^{-12}\text{f}$$

$$=8.9\ \mu\mu\text{f}$$

Thee capacitance with the slab in place is given by

$$C=\frac{K\varepsilon_c A}{Kd-b(K-1)}$$

$$=\frac{(7)(8.9\times 10^{-12}\text{ coul}^2/\text{nt-m}^2)(10^{-2}\text{ meter}^2)}{(7)(10^{-2}\text{ meter})-(0.5\times 10^{-2}\text{ meter})(7-1)}$$

$$=15.6\times 10^{-12}\text{f}=15.6\ \mu\mu\text{f}.$$

(*a*) Charge on the capacitor before the slab is introduced

$$Q_0=C_0V=(8.9\times 10^{-12}\text{f})(100\text{ volt})$$

$$=8.9\times 10^{-10}\text{ coul}$$

Charge on the capacitor after the slab is introduced is

$$Q=CV=(15.6\times10^{-12}\text{f})(100\text{V})=1.6\times10^{-9}\text{ coul.}$$

(*b*) The electric field in the gap before the slab is introduced is

$$E_o=\frac{Q_o}{\varepsilon_o A}=\frac{8.9\times10^{-10}\text{ coul}}{(8.9\times10^{-12}\text{ coul}^2/\text{nt-m}^2)(1.0\times10^{-2}\text{ meter}^2)}$$

$$=1.0\times10^4\text{ volts/meter}$$

The electric field in the gap after the slab is introduced is

$$E'_o=\frac{Q}{\varepsilon_o A}=\frac{1\ 6\times10^{-19}\text{ coul}}{(8.9\times10^{-12}\text{ coul}^2/\text{nt-m}^2)(1.0\times10^{-2}\text{ meter}^2)}$$

$$=1.8\times10^4\text{ volts/meter.}$$

(*c*) The electric field in the slab is given by

$$E=\frac{E_0'}{K}=\frac{1.8\times10^4\text{ volts/meter}}{7}=2.6\times10^3\text{ volts/meter.}$$

30.23. Assume $K=5.4$ for mica

(*b*) The free charge on the plates is

$$q=CV=(100\times10^{-12}\text{ f})(50\text{ volts})=5\times10^{-9}\text{ coul}$$

(*a*) The electric field in the mica is

$$E=\frac{q}{K\varepsilon_o A}$$

$$=\frac{5\times10^{-9}\text{ coul}}{(5.4)(8.9\times10^{-12}\text{ coul}^2/\text{nt-m}^2)(1.0\times10^{-2}\text{ meter}^2)}$$

$$=1.04\times10^4\text{ volts/meter.}$$

(*c*) The induced surface charge is

$$q'=q\left(1-\frac{1}{K}\right)$$

$$=(5\times10^{-9}\text{ coul})\left(1-\frac{1}{5.4}\right)$$

$$=4.1\times10^{-9}\text{ coul}$$

The induced charge of -4.1×10^{-9} coul appears next to the positive plate.

30.24. (*a*) Electric field $E=\dfrac{q}{K\varepsilon_o A}$.

Dielectric constant,

$$K=\frac{q}{\varepsilon_0 EA}$$

$$=\frac{(8.9\times10^{-7}\text{ coul})}{(8.9\times10^{-12}\text{ coul}^2/\text{nt-m}^2)(1.4\times10^{6}\text{ volts/meter})(1.0\times10^{-2}\text{ meter}^2)}$$

$=7.14$

(*b*) Magnitude of the charge induced on each dielectric surface is

$$q'=q\left(1-\frac{1}{K}\right)=(8.9\times10^{-7}\text{ coul})\left(1-\frac{1}{7.14}\right)$$

$=7.7\times10^{-7}$ coul.

30.25. The capacitance of a parallel-plate capacitor is

$$C=\frac{\varepsilon_0 AK}{d} \qquad \ldots(1)$$

Dielectric strength $=18\times10^{6}$ volts/meter.

Electric field strength $=4000$ volts/meter.

Setting dielectric strength equal to the electric field strength, we have

strength we have, $\alpha\ d=\dfrac{4000\text{ volt}}{18\times10^{6}\text{ volts/meter}}=2.22\times10^{-4}$ meter

From (1) we have

$$A=\frac{Cd}{\varepsilon_0 K}=\frac{(7\times10^{-2}\times10^{-6}\text{ f})(2.22\times10^{-4}\text{ meter})}{(8.9\times10^{-12}\text{ coul}^2/\text{nt-m}^2)(2.8)}$$

$=0.62$ meter^2.

30.26. The electric field for a cylinder capacitor is

$$E=\frac{q}{2\pi\varepsilon_0 l r} \qquad \ldots(1)$$

The energy density (energy/unit volume),

$$u=\tfrac{1}{2}\varepsilon_0 E^2=\frac{q^2}{8\pi^2\varepsilon_0 l^2 r^2} \qquad (2)$$

where use has been made of (1).

The energy stored between the coaxial cylinders of length and radius R and a is

$$U=\int u\,dv \quad \int_a^R u\,(2\pi rl)\,dr \qquad \ldots(2)$$

where $dv = (2\pi r\, dr)\, l$, is the volume element.

Using (1) in (2)

$$U = \frac{q^2}{4\pi\, \varepsilon_0\, l} \int_a^R \frac{dr}{r} = \frac{q^2}{4\pi\, \varepsilon_0\, l} \ln \frac{R}{a}$$

Similarly, the energy stored between the coaxial cylinders of radii b and a is

$$U_0 = \frac{q^2}{4\pi\, \varepsilon_0\, l} \ln \frac{b}{a}$$

$$\therefore \quad \frac{U}{U_0} = \frac{\ln R/a}{\ln b/a}$$

Set $\quad \dfrac{U}{U_0} = \dfrac{1}{2}$

Therefore,[3] $\quad \dfrac{\ln R/a}{\ln b/a} = \dfrac{1}{2}$

or $\quad \ln \dfrac{b}{a} = 2 \ln \dfrac{R}{a} = \ln \dfrac{R^2}{a^2}$

and $\quad \dfrac{b}{a} = \dfrac{R^2}{a^2}$

or $\quad R = \sqrt{ab}$

30.27. Radius of the metal sphere $r = 5\text{cm} = 0.05$ meter

Electric field at the surface of the sphere is

$$E = \frac{q}{4\pi\, \varepsilon_0\, r^2}$$

The energy density at the surface is

$$u = \frac{1}{2} \varepsilon_0 E^2 = \frac{q^2}{32\pi^2\, \varepsilon_0\, r^4}$$

But, $\quad q = CV$

$$C = 4\pi\, \varepsilon_0\, r$$

$$\therefore \quad q = 4\pi\, \varepsilon_0\, Vr$$

$$\therefore \quad u = \frac{1}{2} \frac{\varepsilon_0 V^2}{r^2} = \frac{\frac{1}{2}(8.9 \times 10^{-12}\ \text{coul}^2/\text{nt-m}^2)(8000\ \text{volts})^2}{(0.05\ \text{meter})^2}$$

$$= 0.114\ \text{joule/meter}^3$$

30.28. (*a*) Capacity, $C=\frac{\varepsilon_o A}{d}$

Initial potential difference, $V_1=\frac{q}{C}=\frac{q\,d}{\varepsilon_o A}$

New potential difference,

$$V_f=\frac{q\,(2d)}{\varepsilon_o A}=\frac{2qd}{\varepsilon_o A}=2V_1$$

(*b*) Initial stored energy

$$U_i=\frac{1}{2}CV_i^2=\frac{1}{2}\ \frac{\varepsilon_o AV_i^2}{d}$$

Final stored energy

$$U_f=\frac{1}{2}\varepsilon_o.A\ \frac{V_f^2}{d_f}=\frac{1}{2}\varepsilon_o\ A\ \frac{(2V_i)^2}{2d}=2U_i$$

(*c*) Work required to separate the plates

$$W=U_f-U_i=2U_i-U_i=U_i=\frac{\varepsilon_o AV_i^2}{2d}$$

30.29. The energy on the parallel-plate capacitor with plate separation x is

$$U_i=\frac{1}{2}\ \frac{q^2}{C}=\frac{1}{2}\ \frac{q^2}{(\varepsilon_o A/x)}=\frac{1}{2}\ \frac{q^2\,x}{\varepsilon_o A}$$

If the plate separation is increased to $x+dx$, then the energy becomes

$$U_f=\frac{1q^2}{2\varepsilon_o A}(x+dx)$$

Therefore, work to be done to increase the separation of plates through dx is

$$W=U_f-U_i=\frac{1q^2\,dx}{2\varepsilon_o A}=F\,dx$$

$\therefore$ Force $F=\frac{q^2}{2\,\varepsilon_o A}$

30.30. In textbook Example 5, $A=1\times10^{-2}$ meter^2, $d=1.0\times10^{-2}$ meter, $b=0.5\times10^{-2}$ meter, $K=7.0$ and $V_o=100$ volts

For the air-gap energy density, $u_0=\frac{1}{2}\varepsilon_0\ E_0^2=\frac{1}{2}\varepsilon_0\left(\frac{V}{d}\right)^2$

Energy in the air gap, $U_0=u_0\ (\frac{1}{2}dA)$

where $\frac{1}{2}dA$ is the volume corresponding to the air gap.

$$\therefore \quad U_0 = \frac{\varepsilon_0 AV^2}{4d}$$

$$= \frac{1}{4}\frac{(8.9\times10^{-12}\text{ coul}^2/\text{nt-m}^2)(1\times10^{-2}\text{ meter}^2)(100\text{ volts})^2}{(1.0\times10^{-2}\text{ meter})}$$

$$= 2.2\times10^{-8}\text{ joule}$$

For the dielectric, energy density, $u = \frac{1}{K}u_0$.

Since volume is the same as that of air gap,

Energy, $U = \frac{1}{K}\,U_0 = \left(\frac{1}{7}\right)(2.2\times10^{-8}\text{ joules}) = 3\times10^{-9}\text{ joules}$

(*a*) Percentage energy stored in the air gap is

$$\frac{100\,U_0}{U+U_0} = \frac{(100)(2.2\times10^{-8}\text{ joules})}{(0.3\times10^{-8}\text{ joule} + 2.2\times10^{-8}\text{ joule})} = 88\%$$

(*b*) Percentage energy stored in the slab is

$$\frac{100\,U}{U+U_0} = \frac{(100)(0.3\times10^{-8}\text{ joule})}{(0.3\times10^{-8}\text{ joule} + 2.2\times10^{-8}\text{ joule})} = 12\%$$

30.31. (*a*) Energy stored is

$$U = \tfrac{1}{2}CV^2 = \tfrac{1}{2}(100\times10^{-12}\text{ f})(50\text{ volts})^2$$
$$= 1.25\times10^{-7}\text{ joule.}$$

(*b*) Energy density, $u_0 = \frac{\varepsilon_0}{2}\left(\frac{V}{d}\right)^2$

Since the gap d is not known, u_0 cannot be found out.

30.32 (*a*) $U_1 = \frac{1}{2}C_1V_1^2 = \frac{1}{2}(2\times10^{-6}\text{ f})(240\text{ volt})^2 = 5.8\times10^{-2}$ joule.
$U_2 = \frac{1}{2}C_2V_2^2 = \frac{1}{2}(8\times10^{-6}\text{ f})(60\text{ volt})^2 = 1.4\times10^{-2}$ joule.

(*b*) $U_1 = \frac{1}{2}C_1V_1^2 = \frac{1}{2}(2\times10^{-6}\text{ f})(96\text{ volts})^2 = 9.2\times10^{-3}$ joule.
$U_2 = \frac{1}{2}C_2V_2^2 = \frac{1}{2}(8\times10^{-6}\text{ f})(96\text{ volt})^2 = 37\times10^{-3}$ joule.

(*c*) As $V_1 = V_2 = 0$, $U_1 = U_2 = 0$

In (*a*) the capacitors are in series and the energy stored is maximum. In (*b*) the capacitors are in parallel and the energy stored is less. In (*c*) the charges are neutralized and no energy is stored in the capacitors. The energy is used up in heating up the connecting wires.

30.33. The electric energy stored in the soap buble is

$$U_0 = \frac{1}{2}\frac{q^2}{C} = \frac{q^2}{2\,(4\pi\varepsilon_0 R_0)} = \frac{q^2}{8\,\pi\varepsilon_0 R_0}$$

where we have used the formula $C=4\,\pi\varepsilon_0 R_0$ for the capacitance of a spherical capacitor.

Due to mutual repulsion of the charged surface, the radius increases to R leading to a decrease in energy. The new energy is

$$U=\frac{q^2}{8\,\pi\varepsilon_0\,R}$$

$\therefore$ Decrease in energy, $\Delta\, U=U_0-U=\dfrac{q^2}{8\,\pi\varepsilon_0}\left(\dfrac{1}{R_0}-\dfrac{1}{R}\right)$

or $\quad \Delta U=\dfrac{q^2\,(R-R_0)}{8\,\pi\varepsilon_0\,R_0 R}$

Now, the work done in expanding the soap bubble at constant pressure p is

$$\Delta\, W=pdV$$

where dV is the change in volume.

$$dV=\frac{4}{3}\,\pi R^3-\frac{4}{3}\,\pi R_0^3=\frac{4}{3}\,\pi(R^3-R_0^3)$$

Equating,

$$\Delta U=\Delta W$$

$$\frac{q^2\,(R-R_0)}{8\,\pi\varepsilon_0\,R_0 R}=\frac{4}{3}\,\pi\,(R^3-R_0^3)\,p$$

Simplifying, we find

$$q=\left[\frac{32}{3}\,\pi^2\varepsilon_0\,pR_0R\,(R^2+R_0R+R_0^2)\right]^{\frac{1}{2}}$$

$$=\left[\frac{32}{3}\,\pi^2\,(8.9\times10^{-12}\text{ coul}^2/\text{nt-m}^2)\,(1.013\times10^5\text{ nt/m}^2)(0.02\text{ m})\times\right.$$

$$\left.(0.021\text{ m})\left\{(0.02\text{ m})^2+(0.02\text{ m})(0.021\text{ m})+(0.021\text{ m})^2\right\}\right]^{\frac{1}{2}}$$

$$=7.1\times10^{-6}\text{ coul.}$$

30.34. The equivalent capacity of two capacitors in parallel is

$$C=C_1+C_2=2.0\ \mu\text{f}+\mu.0\ \mu\text{f}=6.0\ \mu\text{f}$$

Total energy in the system is

$$U=\tfrac{1}{2}\,CV^2=\tfrac{1}{2}(6\times10^{-6}\text{ f})\,(300\text{ volt})^2=0.27\text{ joule}$$

30.35. The equivalent capacity of n capacitors each of capacitance C_0 in parallel is

$$C=nC_0=(2000)(5.0\times10^{-6}\text{ f})=0.01\text{ f}$$

Energy $U=\frac{1}{2}CV^2=\frac{1}{2}$ (0.01 f) (50,000 volt)^2$=12.5\times10^6$ joule
cost of charging is 2c/kW-hr, or 2c/3.6×10^6 joule

$\therefore$ Cost for charging to the extent of 12.5×10^6 joule is
(2c) (12.5×10^6 joule/3.6×10^6 joule) = 7 c.

30.36. $C_1=10\ \mu f$
$C_2=5\ \mu f$
$C_3=4\ \mu f$
$V=100$ volts

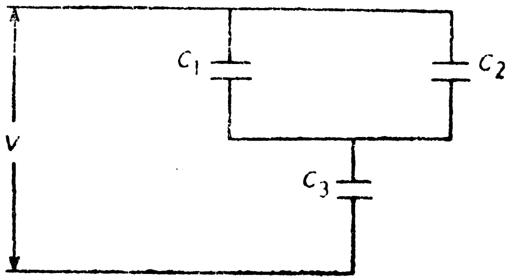


Fig. 30.36

The equivalent capacitance of the arrangement in which C_3 is in series with the combination of C_1 and C_2 in parallel is

$$C=\frac{C_3(C_1+C_2)}{C_1+C_2+C_3}$$
$$=\frac{(4\ \mu f)(10\ \mu f+5\ \mu f)}{(10\ \mu f+5\ \mu f+4\ \mu f)}=3.16\ \mu f$$

The total charge, $q=CV=(3.16\times10^{-6}\ f)\ (100\ volt)=3.16\times10^{-4}$ coul. Charge across C_3 is $q_3=3.2\times10^{-4}$ coul.

(*b*) The potential difference across C_3 is

$$V_3=\frac{q_3}{C_3}=\frac{3.16\times10^{-4}\ coul}{4\times10^{-6}\ f}=79\ volt.$$

It follows that $V_1=V_2=(100-79)$ volt$=21$ volt.

(*a*) $q_1=C_1V_1=(10\times10^{-6}\ f)(21\ volt)=2.1\times10^{-4}$ coul
$q_2=C_2V_2=(5\times10^{-6}\ f)\ (21\ volt)=1.05\times10^{-4}$ coul
$q_3=3.2\times10^{-4}$ coul

(*c*) $U_1=\frac{1}{2}C_1V_1^2=\frac{1}{2}(10\times10^{-6}\ f)(21)^2=2.2\times10^{-3}$ joule
$U_2=\frac{1}{2}C_2V_2^3=\frac{1}{2}(5\times10^{-6}\ f)(21)^2=1.1\times10^{-3}$ joule
$U_3=\frac{1}{2}C_3V_3^2=\frac{1}{2}(4\times10^{-6}\ f)(79)^2=1.25\times10^{-2}$ joule

30.37. $C_1=10\ \mu f$
$C_2=5\ \mu f$
$C_3=4\ \mu f$
$V=100$ volt

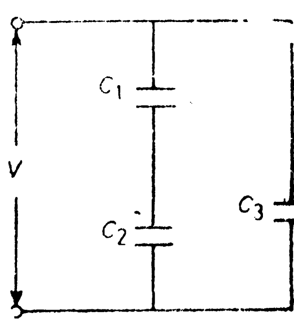


Fig. 30.37

The capacitor C_3 is in parallel to C_1 and C_2 in series. The equivalent capacitance is

$$C=\frac{C_1C_2}{C_1+C_2}+C_3=\frac{(10\ \mu f\ (5\ \mu f)}{(10\ \mu f+5\ \mu f)}+4\ \mu f$$
$$=7.3\ \mu f.$$

The total charge is
$q=CV=(7.3\times10^{-6}\ f)(100\ volt)$
$=7.3\times10^{-4}$ coul.

(*a*) $q_3=C_3V=(4\times10^{-6}\text{ f})(100\text{ volt})=4\times10^{-4}$ coul

$q_1=q_2=q-q_3=(7.3\times10^{-4}-4\times10^{-4})$ coul

$=3.3\times10^{-4}$ coul

(*b*) $$V_1=\frac{q_1}{C_1}=\frac{3.3\times10^{-4}\text{ coul}}{10\times10^{-6}\text{ f}}=33\text{ volt}$$

$$V_2=\frac{q_2}{C_2}=\frac{3.3\times10^{-4}\text{ coul}}{5\times10^{-6}\text{ f}}=67\text{ volt}$$

$V_3=100$ volt

(*c*) $U_1=\frac{1}{2}C_1V_1^2=\frac{1}{2}(10\times10^{-6}\text{ f})(33\text{ volt})^2=5.4\times10^{-3}$ joule

$U_2=\frac{1}{2}C_2V_2^2=\frac{1}{2}(5\times10^{-6}\text{ f})(67\text{ volt})^2=1.1\times10^{-2}$ joule

$U_3=\frac{1}{2}C_3V_3^2=\frac{1}{2}(4\times80^{-6}\text{ f})(100\text{ volt})^2=2\times10^{-2}$ joule.

SUPPLEMENTARY PROBLEMS

S.30.1. Capacitance is given by

$$C=\frac{q}{V}$$

For an isolated sphere of radius *r*,

$$V=\frac{q}{4\pi\varepsilon_0 r}$$

$$\therefore \quad C=4\pi\varepsilon_0 r$$

Let the spheres be oppositely charged.

$$V^+=\frac{q}{4\pi\varepsilon_0 r}$$

$$V^-=-\frac{q}{4\pi\varepsilon_0 r}$$

Potential difference for the system of two spheres

$$V=V^+-V^-=\frac{q}{2\pi\varepsilon_0 r}$$

The corresponding capacitance is then

$$C'=\frac{q}{V}=2\pi\varepsilon_0 r=\frac{1}{2}C.$$

S.30.2. (*a*) Let $Q=Q_a+Q_b$ ...(1)

where the subscripts *a* and *b* refer to spheres of radii *a* and *b*, respectively. The charge will be shared in such a way that the spheres acquire common potential.

$$V=\frac{Q_a}{4\pi\varepsilon_0 a}=\frac{Q_b}{4\pi\varepsilon_0 b} \quad ...(2)$$

whence, $bQ_a=aQ_b$. ...(3)

Solving (1) and (3),

$$Q_a=\frac{aQ}{a+b} \quad ...(3)$$

$$Qb=\frac{bQ}{a+b} \quad ...(4)$$

(*b*) From (2) we have

$$Q_a=4\pi\varepsilon_0 Vb \quad ..(5)$$

$$Q_b=4\pi\varepsilon_0 Va \quad ...(6)$$

Adding (5) and (6)

$$Q_a+Q_b=Q=4\pi\varepsilon_0 V(a+b)$$

$$\therefore \quad C=\frac{Q}{V}=4\pi\varepsilon_0 (a+b)$$

S.30.3. Let q be the charge on each small drop of radius r. Then the potential of each small drop is

$$V_0=\frac{q}{4\pi\varepsilon_0 r} \quad ...(1)$$

The charge on the large drop of radius R is, $Q=Nq$.

The potential of the large drop is

$$V_0=\frac{Nq}{4\pi\varepsilon_0 R} \quad ...(2)$$

Since the large as well as the small drop will have the same density, the volume of the former must be N times as large as the volume of the latter.

$$\frac{4}{3}\pi R^3=N\frac{4}{3}\pi r^3$$

or $$\frac{r}{R}=\frac{1}{N^{1/3}} \quad ...(3)$$

Dividing (2) by (1)

$$\frac{V_0}{V}=N\frac{r}{R}=\frac{N}{N^{1/3}}=N^{2/3} \quad ...(4)$$

where use has been made of (3).

$$\therefore \quad V_0=VN^{2/3}$$

S.30.4. The charges on the two capacitors before and after the switch is closed are shown in Figures S.30.4 (*a*) and (*b*) respectively. As no charge flows through the capacitors, the charge on the plates must be the same before and after the switch is closed.

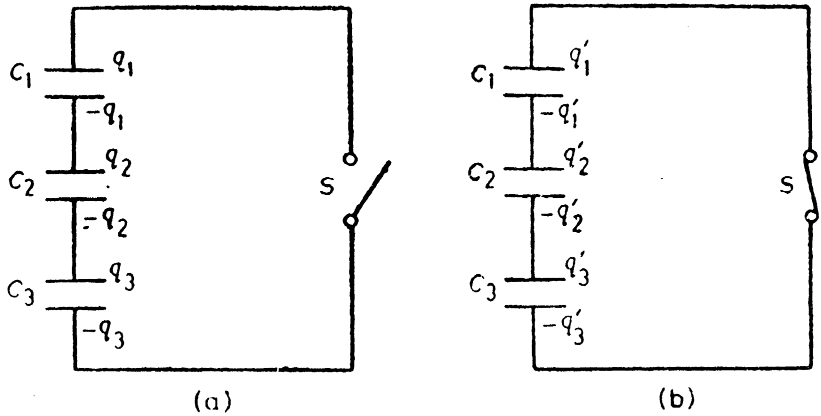


Fig. S-30.4

$$q_2-q_1=q_2'-q_1' \qquad \ldots(1)$$

$$q_3-q_2=q_3'-q_2' \qquad \ldots(2)$$

Further, in the absence of an emf, the voltage across the combination of the capacitors must be zero.

$$\frac{q'}{C_1}+\frac{q_2'}{C_2}+\frac{q_3'}{C_3}=0 \qquad \ldots(3)$$

Solving (1), (2) and (3 for q_1'. q_2' and q_3', we have

$$q_1'=\frac{(C_1C_3+C_1C_2)\,q_1-C_1C_3q_2-C_1C_2q_3}{C_1C_2+C_2C_3+C_3C_1}$$

$$q_2'=\frac{(C_2C_3+C_2C_1)\,q_2-C_2C_3q_1-C_2C_1q_3}{C_1C_2+C_2C_3+C_3C_1}$$

$$q_3'=\frac{(C_3C_1+C_3C_2)\,q_3-C_3C_1q_2-C_3C_2q_1}{C_1C_2+C_2C_3+C_3C_1}$$

S.30.5. Let q_0 be the charge on C_1 when the switch is thrown to the left.

$$q_0=C_1V_0 \qquad \ldots(1)$$

When the switch is thrown to the right as in Fig. S.30.5 the initial charge q_0 is shared among the capacitors such that

$$q_0=q_1+q_2 \qquad \ldots(2)$$

$$q_2=q_3 \qquad \ldots(3)$$

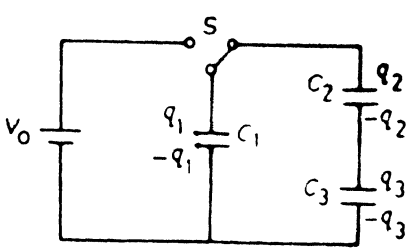


Fig. S.30.5

Further, the potential difference across C_1 is equal to that across the combination of C_2 and C_3.

$$\frac{q_1}{C_1}=\frac{q_2}{C_2}+\frac{q_3}{C_3} \qquad \ldots(4)$$

Solving (2), (3) and (4) for q_1, q_2 and q_3 and using (1),

$$q_1=\frac{(C_2+C_3)\,C_1}{C_1C_2+C_2C_3+C_3C_1}\,C_1V_0$$

$$q_2=q_3=\frac{C_2C_3}{C_1C_2+C_2C_3+C_3C_1}\,C_1V_0$$

S.30.6. If the dielectric is present, Gauss' law gives

$$\varepsilon_0\oint \mathbf{E}.\,d\mathbf{S}=q-q'=\frac{q}{K} \qquad \ldots(1)$$

where $-q'$ is the induced surface charge, q the free charge, and K is the dielectric constant. Construct a Gaussian surface in the form of a coaxial cylinder of radius r and length l, closed by plane caps.

Applying (1),

$$\varepsilon_0E\,(2\pi r)\,(l)=\frac{q}{K} \qquad \ldots(2)$$

The surface contributing to the integral being only the curved surface. and not the end caps. From (2) we find,

$$E=\frac{q}{2\,\pi\varepsilon_0 rlK} \qquad \ldots(3)$$

The potential difference between the central rod and the surrounding tube is given by

$$V=-\int_a^b \mathbf{E}.d\mathbf{r}=\int_a^b E\,dr=\int_a^b\frac{q}{2\,\pi\varepsilon_0\,lK}\,\frac{dr}{r}$$

$$=\frac{q}{2\,\pi\varepsilon_0\,lK}\ln\frac{b}{a}$$

The capacitance is given by

$$C=\frac{q}{V}=\frac{2\,\pi\varepsilon_0 lK}{\ln\,(b/a)}$$

S.30.7. (*a*) If A is the area of the plates and d the distance of separation, then the capacitance before the slab is inserted is found from

$$C_0=\frac{\varepsilon_0\,A}{d}=\frac{\varepsilon_0\,(0.12\ \text{meter}^2)}{1.2\times10^{-2}\ \text{meter}}=10\,\varepsilon_0\ \text{farad}$$

(*d*) $$E_0=\frac{V_0}{d}=\frac{120 \text{ volt}}{1.2\times 10^{-2} \text{ meter}}=10^4 \text{ volt/meter}$$

(*e*) $$E=\frac{E_0}{K}=\frac{10^4 \text{ volt/meter}}{4.8}=2083 \text{ volt/meter}$$

(*f*) $$V=E_0\left[d-b\left(1-\frac{1}{K}\right)\right]$$

$$=(10^4 \text{ volt/meter})\left[0.012 \text{ meter}-0.004 \text{ meter}\left(1-\frac{1}{4.8}\right)\right]$$

$$=88.3 \text{ volt.}$$

(*b*) $$C=\frac{q}{V}=\frac{C_0V_0}{V}=(10\ \epsilon_0)\frac{(120 \text{ volt})}{(88.3 \text{ volt})}=13.6\epsilon_0 \text{ farad}$$

(*c*) Before the slab is inserted,

$$q=C_0V_0=(10\ \epsilon_0 \text{ farad})\ (120 \text{ volt})=1200\ \epsilon_0 \text{ coul.}$$

After the slab is inserted,

$$q'=CV=(13.6\ \epsilon_0 \text{ farad})\ (88.3 \text{ volt})=1200\ \epsilon_0 \text{ coul.}$$

(*g*) $$W=\tfrac{1}{2}\ C_0V_0^2-\tfrac{1}{2}\ CV^2,$$

$$=\tfrac{1}{2}\ (10\epsilon_0 \text{ farad})\ (120 \text{ volt})^2-\tfrac{1}{2}(13.6\ \epsilon_0 \text{ farad})(88.3 \text{ volt})^2$$

$$=1.9\times 10^4\epsilon_0 \text{ joules.}$$

31 CURRENT AND RESISTANCE

31.1. (*a*) Charge flowing=(current) (time)

$$q=it$$

$$=(5 \text{ amp})\ (240 \text{ sec})=1200 \text{ coul}$$

(*b*) Set $q=ne$, where $e=1.6\times10^{-19}$ coul/electron.

where n is the number of electrons flowing, each of charge e.

$$n=\frac{q}{e}=\frac{1200 \text{ coul}}{1.6\times10^{-19} \text{ coul/electron}}=7.5\times10^{21} \text{ electrons.}$$

31.2. Since positive ions and electrons (negative) flow in opposite directions, the effective current in the direction of the flow of positive ions is given by the addition of both the components.

$$i=i_p+i_e$$

$$=\frac{(q_p+q_e)}{t}=\frac{(n_p+n_e)e}{t}$$

$$=(1.1\times10^{18}+3.1\times10^{18})\frac{(1.6\times10^{-19} \text{ coul})}{1.0 \text{ sec}}$$

$$=0.67 \text{ amp}$$

31.3. (*a*) The resistivity ρ may be written as

$$\rho=\frac{V/l}{i/A}$$

For iron and copper wires, V, i and l are the same.

$$\therefore \quad A \propto \rho$$

But $\quad A=\pi r^2$

where r is the radius of cross-section.

$$\therefore \frac{r\ (\text{iron})}{r\ (\text{copper})}=\sqrt{\frac{\rho\ (\text{iron})}{\rho\ (\text{copper})}}=\sqrt{\frac{1.0\times10^{-7} \text{ ohm-m}}{1.7\times10_8^{-} \text{ ohm-m}}}=2.4$$

(*b*) $\quad \rho=\dfrac{V/l}{i/A}=\dfrac{V/l}{j}$

Since V and l are the same, but ρ being different for iron and copper wires, the current densities cannot be made the same with any choice of radii.

31.4.

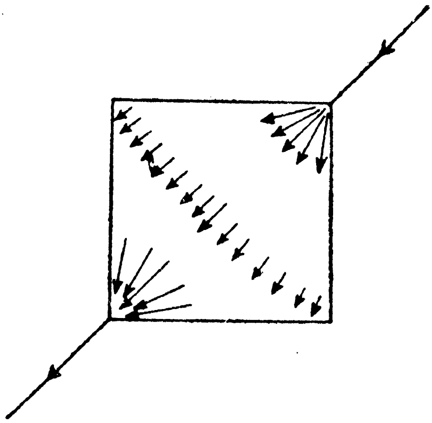


Fig. 31.4

31.5. Let the charge density be σ (coul/meter^2). Charge conveyed by the belt to the sphere per second is

$$i=\frac{q}{t}=\frac{\sigma}{t}(\text{area of belt})=(\sigma)\,(\text{width})(\text{length})/\text{time}$$

$$=(\sigma)\,(\text{width})(\text{speed of belt})$$

$$\therefore \qquad \sigma=\frac{i}{(\text{width})\,(\text{speed})}=\frac{10^{-4}\text{ amp}}{(0.5\text{ meter})(30\text{ meter/sec})}$$

$$=6.7\times10^{-6}\text{ coul/m}^2.$$

31.6. (*a*) Resistance of aluminium rod,

$$R=\frac{\rho l}{A}=\frac{(2.8\times10^{-8}\text{ ohm-m})(1.0\text{ meter})}{(5.0\times10^{-3}\text{ meter})^2}=1.1\times10^{-3}\text{ ohm}$$

(*b*) For the circular copper rod, $A=\frac{\pi}{4}D^2$, where D is the diameter.

$$R=\frac{\rho l}{A}=\frac{\rho l}{\frac{\pi}{4}D^2}$$

Set $\qquad R=1.1\times10^{-3}$ ohm, then

$$D=\sqrt{\frac{4\,\rho\,l}{\pi R}}=\sqrt{\frac{(4)\,(1.7\times10^{-8}\text{ ohm-meter})(1.0\text{ meter})}{(\pi)(1.1\times10^{-3}\text{ ohm})}}$$

$$=4.4\times10^{-3}\text{ meter}=4.4\text{ mm}.$$

31.7. The resistance of the original wire is

$$R_1=\frac{\rho\, l_1}{A_1}=\frac{l_1^2}{A_1\, l_1}=\frac{l_1^2}{v} \qquad \ldots(1)$$

where v is the volume of wire. Since density of the material of wire is constant, it follows that v is also constant.

The resistance of new wire is

$$R_2=\frac{\rho\, l_2^2}{v} \qquad \ldots(2)$$

Dividing (2) by (1)

$$\frac{R_2}{R_1}=\frac{l_2^2}{l_1^2} \qquad \ldots(3)$$

But $\quad l_2=3l_1$

$$\therefore \quad R_2=\frac{l_2^2}{l_1^2}\, R_1=\frac{(3l_1)^2}{l_1^2}\, R_1=9R_1=(9)(6 \text{ ohm})=54 \text{ ohm}$$

31.8. Resistance of copper wire is

$$R_1=\frac{\rho_1 l}{A}=\frac{(1.7\times10^{-8} \text{ ohm-m})(10 \text{ meter})}{\pi\,(1.0\times10^{-3} \text{ meter})^2}=5.4\times10^{-2} \text{ ohm.}$$

Resistance of iron wire is

$$R_2=\frac{\rho_2\, l}{A}=\frac{(1.0\times10^{-7} \text{ ohm-m})(10 \text{ meter})}{\pi\,(1.0\times10^{-3} \text{ meter})^2}=31.8\times10^{-2} \text{ ohm.}$$

(*c*) The potential difference across copper wire is

$$V_1=\frac{VR_1}{R_1+R_2}=\frac{(100 \text{ volt})(5.4\times10^{-2} \text{ ohm})}{(5.4\times10^{-2} \text{ ohm}+31.8\times10^{-2} \text{ ohm})}$$
$$=14.5 \text{ volt}$$

The potential difference across iron wire is

$$V_2=\frac{VR_1}{R_1+R_2}=\frac{(100 \text{ volt})(31.8\times10^{-2} \text{ ohm})}{(5.4\times10^{-2} \text{ ohm}+31.8\times10^{-2} \text{ ohm})}$$
$$=85.5 \text{ volt.}$$

(*a*) Electric field for copper is

$$E_1=\frac{V_1}{l}=\frac{14.5 \text{ volt}}{10 \text{ meter}}=1.45 \text{ volt/meter.}$$

Electric field for iron is

$$E_2=\frac{V_2}{l_2}=\frac{85.5 \text{ volt}}{10 \text{ meter}}=8.55 \text{ volt/meter.}$$

(*b*) Current flowing through the composite wire is

$$i=\frac{V}{R}=\frac{100 \text{ volt}}{(5.4\times10^{-3} \text{ ohm}+31.8\times10^{-3} \text{ ohm})}=2.67 \text{ amp}$$

$\therefore$ Current density, $j=\frac{i}{A}=\frac{2.67 \text{ amp}}{\pi\,(1.0\times10^{-3} \text{ meter})^2}$

$$=8.5\times10^7 \text{ amp/m}^2.$$

31.9. (*b*) $\rho=\frac{AR}{l}=\frac{\pi D^2R}{4l}=\frac{\pi\,(0.55\times10^{-3} \text{ meter})^2(2.87\times10^{-3} \text{ ohm})}{4\,(1.0 \text{ meter})}$

$=6.8\times10^{-8}$ ohm-meter

The materials is nickel (Textbook Table 31.1).

(*a*) Resistance, $R=\frac{\rho l}{\pi r^2}=\frac{(6.8\times10^{-8} \text{ ohm-meter})(1.0\times10^{-3} \text{ meter})}{\pi(1.0\times10^{-3} \text{ meter})^2}$

$=2.17\times10^{-7}$ ohm

31.10. Cross-sectional area $A=7.1 \text{ in}^2=4.58\times10^{-3} \text{ meter}^2$

Length of the track, $l=10 \text{ miles}=1.61\times10^4$ meter

Resistance, $R=\frac{\rho l}{A}=\frac{(6\times10^{-7} \text{ ohm-meter})(1.61\times10^4 \text{ meter})}{4.58\times10^{-3} \text{ meter}^2}$

$=2.1$ ohm.

31.11. (*a*) The resistance R_1 at any temperature T is found from the relation

$$R=R_0\,(1+\alpha\;\Delta T)$$

where R_0 is the resistance at 0°C and α is the temperature coefficient of resistivity. From textbook Table 31.1 we find for copper, $\alpha=3.9\times10^{-3}$.

By Problem, $R=2R_0$

$\therefore$ $2R_0=R_0\,(1+\alpha\;\Delta T)$

$$\Delta T=\frac{1}{\alpha}=\frac{1}{3.9\times10^{-3}}=256°\text{C}$$

At temperature 256°C, the resistance is doubled.

(*b*) As α is independent of shape and size, the result of (*a*) is valid for all copper conductors.

31.12. (*a*) For iron 20° C is

$$\rho=1.0\times10^{-7} \text{ ohm-m and } \alpha=5\times10^{-3}$$

we have, $\rho=\rho_0\,(1+\alpha\Delta T)$

The resistivity of iron at 0°C is

$$\rho_0=\frac{\rho}{1+\alpha\,\Delta\,T}=\frac{1.0\times10^{-7}\text{ ohm-m}}{1+20\times5\times10^{-3}}=0.91\times10^{-7}\text{ ohm-m}$$

For carbon at 20°C

$$\rho'=3.5\times10^{-5}\text{ ohm-m and }\alpha'=-5\times10^{-4}.$$

The resistivity of carbon at 0°C is

$$\rho_0'=\frac{\rho'}{1+\alpha'\Delta T'}=\frac{3.5\times10^{-5}\text{ ohm-m}}{1-20\times5\times10^{-4}}=3.54\times10^{-5}\text{ ohm-m}$$

If the resistance of iron is R and that of carbon R' then the resistance of the composite conductor is

$$R_0=R+R'=\frac{\rho l}{A}+\frac{\rho' l'}{A}$$

where l and l' are the total thickness of iron and carbon disks, respectively. Change in resistance

$$\Delta R_0=\frac{l\,\Delta\rho}{A}+\frac{l'\Delta\rho'}{A}=0$$

$$\therefore\quad \frac{l'}{l}=-\frac{\Delta\rho}{\Delta\rho'}=-\frac{\rho_0\,\alpha\,\Delta T}{\rho_0'\,\alpha'\,\Delta T}=-\frac{\rho_0\,\alpha}{\rho_0'\,\alpha'}$$

$$=-\frac{(0.91\times10^{-7}\text{ ohm-m})(5\times10^{-3})}{(3.54\times10^{-5}\text{ ohm-m})(-5\times10^{-4})}=2.6\times10^{-2}$$

This is also the ratio of thickness of individual disks as the number of disks of either material is the same.

(*b*) As the current is the same in carbon and iron disks, the ratio of the rate of joule-heating in a carbon disk to that in an iron disk is

$$\frac{i^2R'}{i^2R}=\frac{\rho'l'/A}{\rho l/A}=\frac{\rho'l'}{\rho l}=\frac{(3.5\times10^{-5}\text{ ohm-m})(2.6\times10^{-2})}{(1.0\times10^{-7}\text{ ohm-m})}=9.1$$

31.13. The temperature coefficient of resistivity for copper is $3.9\times10^{-3}/°C$.

$\therefore$ Percentage change in R is $(3.9\times10^{-3})\,(100)=0.39\%$.

Coefficient of linear expansion of copper is $1.7\times10^{-5}/°C$.

$\therefore$ Percentage change in length l is

$$(1.7\times10^{-5})\,(100)=0.0017\%$$

Percentage change in area A is $(2)\,(1.7\times10^{-5})(100)=0.0034\%$.

From the above it is seen that percentage changes in l and A are by far less than that in resistance with the variation in temperature.

We, therefore, conclude that for the calculations of variation of resistance with temperature the changes in dimensions can be safely ignored.

31.14. If R_{20}, R_0 and R_T are resistance at 20°C, 0°C and T°C respectively, then

$$R_{20}=R_0\,(1+20\,\alpha) \qquad \ldots(1)$$

$$R_T=R_0\,(1+T\,\alpha) \qquad \ldots(2)$$

Dividing (2) by (1)

$$\frac{R_T}{R_{20}}=\frac{1+T\,\alpha}{1+20\,\alpha}$$

For copper, the temperature coefficient of resistivity is

$$\alpha=3.9\times 10^{-3} \text{ per °C}$$

$$\frac{R_T}{R_{20}}=\frac{58 \text{ ohm}}{50 \text{ ohm}}=\frac{1+(3.9\times 10^{-3})\,T}{1+20\times 3.9\times 10^{-3}}$$

Solving, we find the temperature, $T=64$°C.

31.15. The resistance is found from the relation, $R=\dfrac{i}{V}$

The plot of R against V for the vacuum tube is shown in Fig. 31.15 (*a*) and for the termister in Fig 31.15 (*b*).

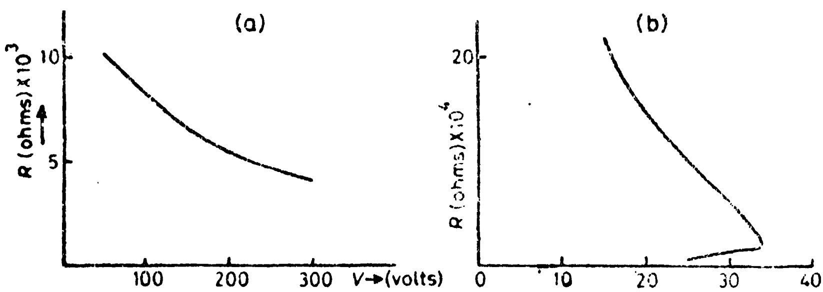


Fig. 31.15

31.16. The drift velocity is given by

$$v_d=\frac{j}{ne}$$

Where j is the current density, n is the number of electrons/cm³ and e the electron charge.

$$j=\frac{i}{A}=\frac{10^{-10}\text{ amp}}{\pi\,(0.127\text{ cm})^2}=1.97\times 10^{-9}\text{ amp/cm}^2$$

If N_0 is Avogadro's number, d the density and M the atomic weight, then number of free electrons/cm^3.

$$n=\frac{N_0 d}{M}=\frac{(6\times10^{23}\text{ atoms/mole})(9.0\text{ gm/cm}^3)(1\text{ electron/atom})}{64\text{ gm/mole}}$$

$$=8.4\times10^{22}\text{ electrons/cm}^3.$$

$$V_d=\frac{j}{ne}=\frac{1.97\times10^{-1}\text{ amp/cm}^2}{(8.4\times10^{22}\text{ electrons/cm}^3)(1.7\times10^{-19}\text{ coul})}$$

$$=1.5\times10^{-13}\text{ cm/sec}.$$

31.17. Rate of energy transfer is,

$$P=i^2R$$

$$R=\frac{P}{i^2}=\frac{(100\text{ watts})}{(3.0\text{ amp})^2}=11\text{ ohms}.$$

31.18. Length, $l=100$ ft$=30.48$ meter

Radius, $r=0.02$ in$=5.1\times10^{-4}$ meter

Cross-section area, $A=\pi r^2=\pi\,(5.1\times10^{-4}\text{ meter})^2$

$$=8.17\times10^{-7}\text{ meter}^2$$

Resistivity for copper, $\rho=1.7\times10^{-8}$ ohm-m.

(*a*) Resistance, $R=\dfrac{\rho l}{A}=\dfrac{(1.7\times10^{-8}\text{ ohm-m})(30.48\text{ meter})}{(8.17\times10^{-7}\text{ meter}^2)}$

$$=0.63\text{ ohm}.$$

Current, $i=\dfrac{V}{R}=\dfrac{1.0\text{ volt}}{0.63\text{ ohm}}=1.6$ amp.

(*b*) The current density, $j=\dfrac{i}{A}=\dfrac{1.6\text{ amp}}{8.17\times10^{-7}\text{meter}^2}$

$$=1.96\times10^6\text{ amp/meter}^2$$

(*c*) Electric field strength, $E=\dfrac{V}{l}=\dfrac{1.0\text{ volt}}{30.48\text{ meter}}$

$$=3.3\times10^{-2}\text{ volt/meter}$$

(*d*) The rate of joule-heating, $P=\dfrac{V^2}{R}=\dfrac{(1.0\text{ volt})^2}{0.63\text{ ohm}}=1.6$ watts.

31.19. Radius of wire $r=0.05$ in$=1.27\times10^{-3}$ meter.

Cross-section area of wire, $A=\pi r^2=\pi\,(1.27\times10^{-3}\text{ meter})^2$

$$=5.06\times10^{-6}\text{ meter}^2.$$

Length of the wire, $l=1000$ ft$=304.8$ meter

(*a*) Current density, $j=\frac{i}{A}=\frac{25 \text{ amp}}{5.06\times10^{-6} \text{ meter}^2}$

$=4.94\times10^6$ amp/meter²

(*b*) Electric field strength, $E=\rho j$

$=(1.7\times10^{-8}$ ohm-m$)(4.94\times10^6$ amp/meter²$)$

$=8.4\times10^{-2}$ volt/meter

(*c*) Potential difference, $V=El$

$=(8.4\times10^{-2}$ volt/meter$)(304.8$ meter$)=25.6$ volt

(*d*) Rate of joule heating, $P=Vi$

$=(25.6$ volt$)$ $(25$ amp$)$ $=640$ watts

31.20. (*a*) Heat absorbed per second$=(500$ joules$)(80\%)$

$=400$ joules

$=\frac{400 \text{ joules}}{4.186 \text{ joules/cal}}=95.6$ cal

Heat required to raise the temperature of water from 20°C to 100°C is

$H=mc\,\Delta T$

$=(2000$ gm$)$ $(1.0$ cal/gm$)$ $(100°C-20°C)=16\times10^4$ cal.

Time required to bring the water to boiling temperature

$=\frac{\text{Heat absorbed by water}}{\text{Rate of absorption of heat}}$

$=\frac{16\times10^4 \text{ cal}}{95.6 \text{ cal/sec}}=1670$ secs$=28$ minutes.

(*b*) Heat required to boil half the water i.e., 1 liter or 1000 gm,

$H=mL=(1000$ gm$)$ $(540$ cal/gm$)$

$=54\times10^4$ cal.

Time taken to boil half the water away

$=\frac{\text{Heat required for vapourization}}{\text{Rate of absorption of heat}}$

$=\frac{54\times10^4 \text{ cal}}{95.6 \text{ cal/sec}}=5648$ sec$=94$ minutes.

31.21. Power, $P=\frac{V^2}{R}$

Resistance at 800° C, $R_{800}=\frac{V^2}{P}=\frac{(110 \text{ volt})^2}{500 \text{ watt}}=24.2$ ohm

Resistance at 0°C

$$R_0=\frac{R_{800}}{1+\alpha\ \Delta T}=\frac{24.2\text{ ohm}}{1+(4\times10^{-4}/°\text{C})(800°\text{C})}=18.33\text{ ohm}$$

Resistance at 200°C, $R_{200}=R_0\ (1+\alpha\ \Delta\ T)$

$$=(18.33\text{ ohm})\ [1+(4\times10^{-4}/°\text{C})(200°\text{C})]=19.8\text{ ohm}$$

Power dissipated at 200°C,

$$P'=\frac{V^2}{R_{200}}=\frac{(110\text{ volt})^2}{19.8\text{ ohm}}=611\text{ watts}$$

31.22. (*a*) Current, $i=ne$,

Where n is the number of deutrons of charge e striking the block.

$$n=\frac{i}{e}=\frac{15\times10^{-6}\text{ amp}}{1.6\times10^{-19}\text{ coul}}=9.4\times10^{13}\text{ deutrons per sec.}$$

If the deutrons are completely absorbed in the block then the energy dissipated per sec$=nK$, where K is the kinetic energy of each deutron.

Total energy available per sec

$$=nK$$
$$=(9.4\times10^{13}/\text{sec})\ (16\text{ Mev})$$
$$=1.5\times10^{15}\text{ Mev/sec}$$
$$=1.5\times10^{21}\text{ Mev/sec}$$

$\therefore$ Heat evolved/sec$=(1.5\times10^{21}\text{ eV/sec})(1.6\times10^{-19}\text{ joule/eV})$

$$=240\text{ joule/sec}$$
$$=240\text{ watts.}$$

31.23. (*a*) Power, $P=\frac{V^2}{R}$

$\therefore$ Resistance, $R=\frac{V^2}{P}=\frac{(115\text{ volt})^2}{500\text{ watt}}=26.45\text{ ohm}$

Power is altered to $P'=\frac{(V')^2}{R}=\frac{(110\text{ volt})^2}{(26.45\text{ ohm})}=457\text{ watts}$

Percentage drop in power,

$$\frac{\Delta P}{P}\times100=\frac{(500-457)\text{ watts}}{500\text{ watts}}=8.6\%.$$

(*b*) With the drop in power, temperature as well as resistance would decrease. Therefore, the actual heat output P' will be larger and so $\frac{\Delta P}{P}$ will be smaller than that calculated in (*a*).

31.24. Power $P_0=i^2R=i^2\rho\dfrac{l}{A}$

Power per unit volume,

$$p=\frac{P_0}{V}=\frac{i^2\rho l/A}{lA}=\frac{i^2\rho}{A^2}=j^2\rho$$

where $V=lA$ is the volume.

Also $\rho=\dfrac{E}{j}$ i.e. $j=\dfrac{E}{\rho}$

$\therefore$ $p=E^2/\rho$

SUPPLEMENTARY PROBLEMS

S.31.1. (*a*) Let N α-particles strike the plane surface in time t secs. Then the current,

$$i=\frac{Nq}{t}=\frac{N\,(2e)}{t}$$

$$\therefore \quad N=\frac{it}{2e}=\frac{(0.25\times10^{-6}\text{ amp})(3\text{ secs})}{(2)\,(1.6\times10^{-19}\text{ coul})}=2.3\times10^{12}$$

(*b*) Speed of α-particles,

$$v=\sqrt{\frac{2K}{m}}=\sqrt{\frac{(2)\,(20\times10^{6}\text{ eV})(1.6\times10^{-19}\text{ joule/eV})}{6.68\times10^{-27}\text{ kg}}}$$

$$=3.1\times10^{7}\text{ meter/sec.}$$

Number of α-particles crossing a given plane area per second,

$$n=\frac{i}{q}=\frac{0.25\times10^{-6}\text{ amp}}{(2)\,(1.6\times10^{-19}\text{ coul})}$$

$$=7.8\times10^{11}\ \alpha\text{-particls/sec.}$$

Number of α-particles per meter length of the beam

$$=\frac{n}{v}=\frac{7.8\times10^{11}/\text{see}}{3.1\times10^{7}\text{ meter/sec}}=2.5\times10^{4}/\text{per meter}$$

Hence number of α-particles per 0.2 meter length of the beam

$$=(2.5\times10^{4}\text{ per meter})(0.2\text{ meter})=5000\ \alpha\text{-particles.}$$

(*c*) By definition a particle of charge ze in falling through a potential difference of 1 volt gains energy equal to z electron volt. For α-particle $z=2$, hence, the potential required to accelerate α-particles to 20 Mev is

$$V=\frac{20\times10^{6}\text{ eV}}{2}=10^{7}\text{ volts.}$$

S.31.2. Table listing similarities and differences

Item	Flow of charge	Flow of fluid	Conduction of heat
(a) Direction of flow	Higher electrical potential to a lower one	Higher pressure head to lower one	Higher temperature to lower one
(b) Condition for steady flow	Constant potential difference required.	Constant pressure heed required.	Constant temperature difference required
(c) Rate of flow	Rapid	Slow	Slow
(d) Mechanism of flow	Free electrons in metals, ions and electrons in electrolytes.	Molecules	Electrons and molecules
(e) Equation of continuity	$\Delta . j+\frac{\partial \rho}{\partial t}=0$ Where j is current density and ρ charge density	$\nabla . S+\frac{\partial \rho}{\partial t}=0$ S is fluid current and ρ fluid density	$\nabla . \phi+c\rho\frac{\partial T}{\partial t}=0$ ϕ is heat flux, T temperature c is specific heat and ρ density of material
(f) Conservation law	Conservation of charge	Conservation of mass	Conservation of heat
(g) Surface/volume transfer of energy	Mainly surface	Within interior	Within interior

Item	Flow of charge	Flow of fluid	Conductiod of heat
(*h*) Friction	Electrical resistance opposes flow of charge and converts electrical energy into heat	Viscosity opposes flow of fluid	Thermal resistance opposes conduction of heat
(*i*) Equation for current	$i=\sigma A \frac{dV}{dx}$	$V=\frac{(\pi R^4)(p_1-p_2)}{8\eta L}$	$H=KA \frac{dT}{dx}$
	i is the electric current, σ the electric conductivity, A the area of cross-section, and dV/dx the potential gradient.	Where V is the volume of fluid crossing any section of a horizontal pipe of radius R and length L, p_1-p_2 the pressure difference between the ends of the tube and in the viscosity of the fluid	Where H is the heat current, K the thermal conductivity A the cross-section area, and the temperature gcadient.
(*j*) Suitable material for energy transfer	Metals alloys and electrolytes	Fluids	Metals and alloys

S.31.3. As the conduction electrons get accelerated, sooner or later, they suffer loss of energy following collisions with the ions in the lattice of the metal. The kinetic energy of the electrons is dissipated into heat by way of imparting energy to the lattice vibrations. The conduction electrons being part of the conductor can only exert internal force on the conductor and therefore this can not give rise to a resultant force on the conductor.

S.31.4. (*a*) Consider an elementary disk of radius r and of thickness dx at distance x from the truncated end and symmetrical about the axis. Then,

$$r=a+\frac{(b-a)}{l}x \qquad \ldots(1)$$

The resistance for this volume-element is

$$dR=\frac{\rho\,dx}{\pi r^2}=\frac{\rho l^2 dx}{\pi[la+(b-a)x]^2} \qquad \ldots(2)$$

where use has been made of (1).

$$R=\int dR=\int_0^l \frac{\rho l^2 dx}{\pi\;[la+(b-a)x]^2}$$

Set, $\quad y=la+(b-a)x$

$$dy=(b-a)dx$$

So that,

$$R=\frac{\rho l^2}{\pi(b-a)}\int_{al}^{bl}\frac{dy}{y^2}=\frac{\rho l^2}{\pi(b-a)}\left(\frac{1}{al}-\frac{1}{bl}\right)$$

Simplifying,

$$R=\frac{\rho l}{\pi ab} \qquad \ldots(3)$$

(*b*) For $a=b$, (3) becomes

$$R=\frac{\rho l}{\pi a^2}=\frac{\rho l}{A}$$

with $\quad A=\pi a^2$.

S.31.5. (*a*) The resistivities of the two wires are given by

$$\rho_A=\frac{R_A\,A}{l}=\frac{(40\text{ ohm})(0.1\text{ meter}^2)}{(40\text{ meter})}=0.1\text{ ohm-meter.}$$

$$\rho_B=\frac{R_B A}{l}=\frac{(20\text{ ohm})(0.1\text{ meter}^2)}{(40\text{ meter})}=0.05\text{ ohm-meter.}$$

(*b*) and (*d*) : The resistance of the two wires in series is

$$R = R_A + R_B = 40 \text{ ohm} + 20 \text{ ohm} = 60 \text{ ohm}.$$

Current, $i = \dfrac{V}{R} = \dfrac{60 \text{ volt}}{60 \text{ ohm}} = 1.0$ amp

The potential difference across wire A is

$$V_A = iR_A = (1.0 \text{ amp})(40 \text{ ohm}) = 40 \text{ volt}.$$

Therefore, the field is $E_A = \dfrac{V_A}{l} = \dfrac{40 \text{ volt}}{40 \text{ meter}} = 1.0$ volt/meter

The potential difference across wire B is

$$V_B = iR_B = (1.0 \text{ amp})(20 \text{ ohm}) = 20 \text{ volt}$$

Therefore, the field is $E_B = \dfrac{V_B}{l} = \dfrac{20 \text{ volt}}{40 \text{ meter}} = 0.5$ volt/meter.

(*c*) The current density in each wire is

$$j_a = j_b = \frac{1.0 \text{ amp}}{0.1 \text{ meter}^2} = 10 \text{ amp/meter}^2.$$

S.31.6. (*a*) Power, $P = Vi$

$\therefore$ Current, $i = \dfrac{P}{V} = \dfrac{1250 \text{ watt}}{115 \text{ volt}} = 10.87$ amp.

(*b*) Resistance, $R = \dfrac{V}{i} = \dfrac{115 \text{ volt}}{10.87 \text{ amp}} = 10.6$ ohm

(*c*) Power, $P = 1250$ watt $= 1.25$ kilo watt.

In 1 hour 1.25 kw-hr energy produced is

$$1.25 \text{ kw-hr} = (1.25 \text{ kw-hr}) \frac{(860 \text{ kcal})}{(1 \text{ kw-hr})} = 1075 \text{ kcal}$$

S.31.7. At equilibrium temperature,

Rate of heat radiated = Joule heating

$$\sigma(T^4 - T_0^4)(2\pi rl) = i^2 R \qquad \ldots(1)$$

Where T is the temperature of iron wire, $T_0 = (273+27) = 300$ °K that of surroundings, $2\pi rl$ = surface area of wire (neglecting the area of cross-section of wire at the ends), $i = 10$ amp is the current, R the resistance, $\sigma = 5.67 \times 10^{-8}$ watt/(meter^2)(°K^4), and $r = 0.5 \times 10^{-3}$ meter is the radius of wire.

$$R = \frac{\rho_T l}{\pi r^2} \qquad \ldots(2)$$

$$\rho_T = \rho_0[1 + \alpha(T - T_0)] \qquad \ldots(3)$$

At 20°C, $\rho_{(T_0+20)}=\rho_0(1+20\,\alpha)$ ...(4)

Divinding (3) by (4),

$$\frac{\rho_T}{\rho(T_0+20)}=\frac{1+\alpha\,(T-T_0)}{1+20\alpha}$$

or $$\rho_T=\frac{\rho\,(T_0+20)}{1+20\alpha}\,[1+\alpha\,(T-T_0)] \quad ...(5)$$

Combining (1), (2) and (5), and using

$\rho_{(T_0+20)}=10^{-7}$ ohm-m and $\alpha=5\times10^{-3}/°C$

and solving for T, we find $T=670°K$

$=(670-273)°C$

$=400°C$.

23 ELECTROMOTIVE FORCE AND CIRCUITS

32.1. Resistance $R=\frac{V}{i}=\frac{6.0 \text{ volt}}{5.0 \text{ amp}}=1.2$ ohm

Chemical energy is reduced due to joule-heating and is given by

$$Pt=i^2Rt=(5 \text{ amp})^2 \ (1.2 \text{ ohm}) \ (360 \text{ secs})$$
$$=1.08\times 10^4 \text{ joules}$$

32.2. Let the resistance of the original circuit be R and a potential difference V be applied. Then

$$\frac{V}{R}=i=5 \text{ amp}$$

With the additional resistance of 2 ohm the current drops to 4 amp

$$\therefore \quad \frac{V}{R+2}=4 \text{ amp}$$

Dividing the two equations,

$$\frac{R+2}{R}=\frac{5}{4}, \text{ whence } R=8 \text{ ohm.}$$

32.3. Potential difference,

$$V=i_1R=(5 \text{ amp})\ (8 \text{ ohm})=40 \text{ volt.}$$

The new resistance of circuit is $R_2=(8+0.05)$ ohm or 8.05 ohm. The current drops to

$$i_2=\frac{V}{R_2}=\frac{40 \text{ volt}}{8.05 \text{ ohm}}=4.969 \text{ amp.}$$

$\therefore$ Change in current $\Delta i=i_2-i_1=(4.969-5.0)=-0.031$ amp

$\therefore$ Percent change in current, $\frac{\Delta i}{i_1}\times 100=\left(\frac{-0.031 \text{ amp}}{5.0 \text{ amp}}\right)\times 100$

$=-0.62\%$.

32.4. (a) Current, $i=\frac{E}{R+r}$

Set $E=2.0$ volt and $r=100$ ohms, Fig. 32.4 (a). Then

$$i=\frac{2}{R+100} \qquad \ldots(1)$$

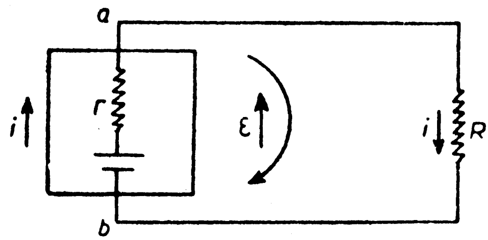


Fig. 32.4 (a)

Curve (*a*) in Fig. 32.4 (*b*) shows the plot of current i as a function of R over the range 0 to 500 ohms.

(*b*) The potential difference across the resistor R is

$$V = iR = \frac{2\,R}{R+100}$$

where use has been made of (1).

Curve (*b*) in Fig. 32.4 (*b*) shows the plot of potential V across the resistor R as a function of R.

(*c*) Curve (*c*) in Fig 32.4 (*b*) shows the plot of the product $P = Vi$ as a function of R. This plot shows the variation of power P with the external resistance R.

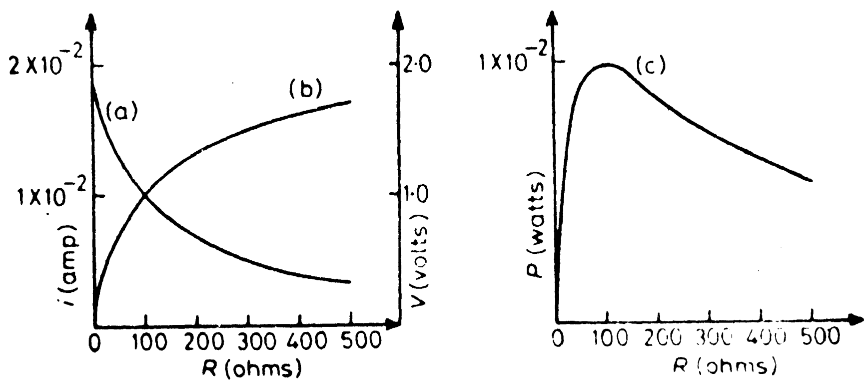


Fig. 32.4 (b)

Curve (*c*) in Fig. 32.4 (*b*) corresponds to the plot of power P delivered to the resistor R as a function of R. It is seen that P has maximum value at $R = 100$ ohms, a value which is identical with r, the internal resistance of the battery.

32.5. (*a*) Current, $i = \dfrac{E}{R+r}$ ...(1)

Power delivered, $P = i^2 R = \dfrac{E^2 R}{(R+r)^2}$ ...(2)

For maximum power set $\frac{\partial P}{\partial R}=0$.

$$\frac{\partial P}{\partial R}=\frac{E^2 dP/dR}{(R+r)^2}+E^2R\,\frac{d}{dR}(R+r)^{-2}=0$$

$$\therefore\ \frac{1}{(R+r)^2}-\frac{2\,R}{(R+r)^3}=0$$

whence, $R=r$

(*b*) Maximum power is obtained by putting $R=r$ in (2).

$$P_{max}=\varepsilon^2/4\,r$$

32.6. (*a*) Current, $i=\dfrac{E}{(R+r)}$

Power $P=i^2R=\dfrac{E^2\,R}{(R+r)^2}$

$$r=\sqrt{E^2\,R/P}-R$$

$$=[(1.5\text{ volt})^2\,(0.1\text{ ohm})/(10\text{ watts})]^{1/2}-0.1\text{ ohm}$$

$$=0.05\text{ ohm.}$$

(*b*) Potential difference across the resistor,

$$V=iR=R\sqrt{P/R}=\sqrt{PR}=\sqrt{(10\text{ watt})\,(0.1\text{ ohm})}=1\text{ volt.}$$

32.7. (*a*) E_1 and E_2 are in opposition. Effective emf is given by

$$E=E_2-E_1$$

Current, $i=\dfrac{E_2-E_1}{R+r+r}=\dfrac{E_2-E_1}{R+2r}$

$$0.001\text{ amp}=\frac{(3-2)\text{ volt}}{(R+6)\text{ ohm}}$$

whence, $R=994$ ohm.

(*b*) Rate of joule heating in R is

$$P=i^2R=(0.001\text{ amp})^2\,(994\text{ ohms})$$

$$=9.94\times10^{-4}\text{ watts.}$$

32.8. (*a*) Current, $i=\dfrac{E}{R+r}$

Power developed in the resistance $(R+r)$ is

$$P_e=i^2\,(R+r)=\frac{E^2}{R+r}=\frac{(2.0\text{ volt})^2}{(5+1)\text{ ohm}}=\frac{2}{3}\text{ watt.}$$

Energy transferred from chemical to electrical form is

$$P_0t=\left(\frac{2}{3}\text{ watt}\right)(120\text{ sec})=80\text{ joules.}$$

(*b*) Joule heating in the wire is

$$P=i^2R=\frac{E^2R}{(R+r)^2}=\frac{(2.0\text{ volt})^2(5\text{ ohm})}{(5\text{ ohm}+1\text{ ohm})^2}=\frac{5}{9}\text{ watt.}$$

Total energy that appears in the wire as joule heat is

$$Pt=\left(\frac{5}{9}\text{ watt}\right)(120\text{ sec})=66.7\text{ joule.}$$

(*c*) The difference in energy $(80-66.7)$ or 13.3 joules is to be attributed to the joule heating of the battery owing to its internal resistance.

32.9. $E_1=4$ volt
$E_2=1$ volt
$R_1=R_2=10$ ohm
$R_3=5$ ohm

Traversing the right loop in the clockwise direction,

$$E_2+i_2R_2+i_3R_3=0 \qquad \ldots(1)$$

Traversing the left loop in the counter clockwise direction,

$$E_1-i_1R_1+i_3R_3=0 \qquad \ldots(2)$$

Traversing the path *b a d c b* in the counter clockwise sense,

$$E_1-i_1R_1-i_2R_2-\mathcal{E}_2=0 \qquad \ldots(3)$$

The junction theorem yields

$$i_1+i_3-i_2=0 \qquad \ldots(4)$$

It is obvious that (3) is not an independent equation as it can be obtained by substracting (1) from (2). Substituting the numerical values in (1), (2) and (4) we have

$$10\,i_2+5\,i_3=-1 \qquad (1)'$$

$$10\,i_1-5\,i_3=4 \qquad (2)'$$

$$i_1-i_2+i_3=0 \qquad (4)'$$

Solving (1)', (2)' and (4)'

$$i_2=0.025\text{ amp}$$

$$\therefore \quad V_{cd}=i_2R_2=(0.025\text{ amp})\,(10\text{ ohm})=0.25\text{ volt.}$$

Traversing various loops in opposite sense would not yield any new information.

32.10. $E_1=2$ volt
$E_2=4$ volt
$r_1=1$ ohm
$r_2=2$ ohm
$R=5$ ohm

Start from c and traverse in the counter clockwise direction along b and thence to a.

$$V_{ac}=-i\,(R+r_2)+E_2$$
$$=-i(5+2)+4$$
$$=(-7\,i+4)\text{ volt} \qquad \ldots(1)$$

Applying loop theorem to the entire circuit starting from c and going in the counter clockwise direction

$$-i\,(R+r_1+r_2)+E_2-E_1=0$$

$$\therefore \qquad i=\frac{E_2-E_1}{R+r_1+r_2}=\frac{(4-2)\text{ volt}}{(5+1+2)\text{ ohm}}=0.25\text{ amp}$$

Using this value of i in (1),

$$V_{ac}=-(7\text{ ohm})\,(0.25\text{ amp})+4\text{ volt}$$
$$=2.25\text{ volt}$$

32.11. (*a*) The resistors R_2, R_3 and R_4 are in parallel. The equivalent resistance R_5 for these three resistors is given by

$$\frac{1}{R_5}=\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}$$
$$=\frac{1}{50}+\frac{1}{50}+\frac{1}{75}=\frac{4}{75}$$
$$R_5=\frac{75}{4}=18.75\text{ ohm}$$

The equivalent resistance R of the network is obtained by combining R_5 and R_1 in series

$$R=R_5+R_1=18.75+100=118.75\text{ ohm.}$$

(*b*) Let current i_1, i_2, i_3 and i_4 flow through resistors R_1, R_2, R_3 and R_4, respectively. Applying junction theorem at the junction of all the four resistances

$$i_1=i_2+i_3+i_4 \qquad \ldots(1)$$

Applying the loop theorem for the path comprising R_1 and R_2,

$$E-i_1R_1-i_2R_2=0 \qquad \ldots(2)$$

Applying the loop theorem for the path comprising R_3 and R_4,

$$-i_3R_3+i_4R_4=0 \qquad \ldots(3)$$

Applying the loop theorem for the path comprising R_2 and R_4,

$$-i_4R_4+i_2R_2=0 \qquad \ldots(4)$$

Put $R_1=100$ ohms, $R_2=R_3=50$ ohms, $R_4=75$ ohms and $E=6$ volts, and solve the simultaneous equations (1), (2), (3) and (4) to obtain

$$i_1=0.0505 \text{ amps}$$
$$i_2=i_3=0.0189 \text{ amps}$$
$$i_4=0.0126 \text{ amps.}$$

We can get the result for (*a*) by an alternative method. Since current i_1 gets distributed through various resistors of the circuit the equivalent resistance of the entire circuit is

$$R=\frac{E}{i_1}=\frac{6 \text{ volts}}{0.0505 \text{ amps}}=118.8 \text{ ohms}$$

32.12. (*a*) Let the current i_1, i_2 and i_3 flow through R_1, R_2 and R_3 respectively.

Applying the junction theorem at the junction of R_1, R_2 and R_3,

$$i_1=i_2+i_3 \qquad \ldots(1)$$

Applying the loop theorem to the left loop,

$$E-i_2R_2-i_1R_1=0 \qquad \ldots(2)$$

Applying the loop theorem to the right loop,

$$i_2R_2-i_3R_3=0 \qquad \ldots(3)$$

Put $R_1=2$ ohm, $R_2=4$ ohm, $R_3=6$ ohm and $E=5$ volts, and solve the simultaneous equations (1), (2) and (3) to get the current in the ammeter $i_3=0.45$ amps.

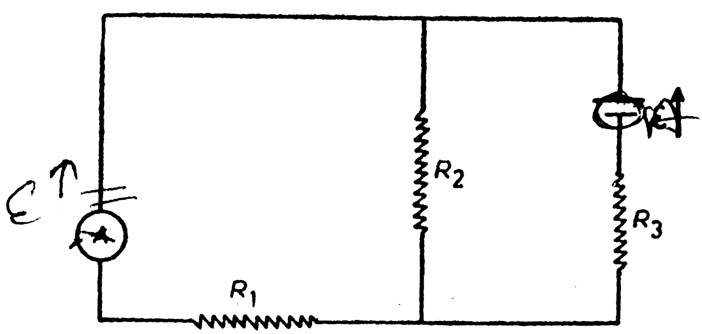


Fig. 32.12 (a)

(*b*) Junction theorem for the circuit in Fig. 32.12 (*a*) yields

$$i_3=i_1+i_2 \qquad \ldots(4)$$

Applying loop theorem to the left loop

$$i_1R_1 - i_2R_2 = 0 \qquad \ldots(5)$$

Applying the loop theorem to the right

$$E - i_2R_2 - i_3R_3 = 0 \qquad \ldots(6)$$

Using the numerical values of (*a*) we find,

$$i_1 = 0.45 \text{ amp.}$$

This represents the current indicated by the ammeter. Thus, the ammeter reading remains unchanged.

32.13. Applying the junction theorem to the junction of the three resistors,

$$i_1 + i_2 = i \qquad \ldots(1)$$

Applying the loop theorem to the lower loop,

$$E - i_2r - iR = 0 \qquad \ldots(2)$$

Applying the loop theorem to the big loop

$$E - i_1r - iR = 0 \qquad \ldots(3)$$

Eliminating i_1 and i_2, we have

$$i = \frac{E}{R + r/2} \qquad \ldots(4)$$

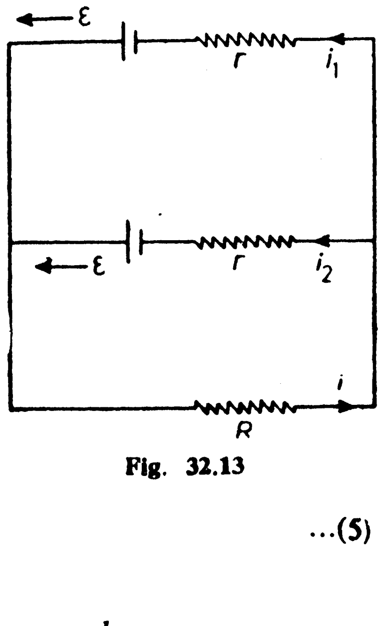


Fig. 32.13

(*a*) Power delivered to the resistor *R*,

$$P = i^2R = \frac{E^2R}{\left(R + \frac{r}{2}\right)^2} \qquad \ldots(5)$$

Maximum value of P is obtained by setting $\frac{dp}{dR} = 0$

$$\frac{dP}{dR} = \frac{E^2\left(R + \frac{r}{2}\right)^2 \frac{dR}{dR} - E^2 R \frac{d}{dR}\left(R + \frac{r}{2}\right)^2}{\left(R + \frac{r}{2}\right)^4} = 0$$

or $$\left(R + \frac{r}{2}\right)^2 - 2R\left(R + \frac{r}{2}\right) = 0$$

whence $$R = \frac{r}{2} \qquad \ldots(6)$$

(*b*) Use (6) in (5) to find

$$P_{(max)} = \frac{E^2}{2r}$$

32.14. Let the resistances be R_1 and R_2

In series, $R_1+R_2=R_3$ ...(1)

In parallel, $R_4=\dfrac{R_1R_2}{R_1+R_2}=\dfrac{R_1R_2}{R_3}$

where use has been made of (1).

$\therefore \quad R_1R_2=R_3R_4$ ...(2)

Of the resistances 3, 4, 12 and 16 ohms the choice $R_1=4$ ohms, $R_2=12$ ohm when used in series or parallel would satisfy both (1) and (2) and therefore provide the equivalent resistance 16 ohm and 3 ohm respectively. When used singly they provide the resistances of 4 ohms and 12 ohms.

32.15. Suppose that *a* and *b* are at the same potential.

If i_1 is the current flowing through R_1 and i_s through R_s, then the potential drop over R_1 must be identical with that over R_s.

$i_1R_1=i_sR_s$ ...(1)

Similarly, the potential drop over R_2 must be the same as that over R_x.

$i_2R_2=i_sR_x$ ...(2)

Dividing (1) by (2),

$\dfrac{R_1}{R_2}=\dfrac{R_s}{R_x}$ or $R_x=R_s\dfrac{R_2}{R_1}$.

32.16. Applying the junction theorem at *b*, (Fig 32.16).

$i_s+i=i_x$ ...(1)

Applying the junction theorem at *a*,

$i_2+i=i_1$ ...(2)

Applying the loop theorem to the path *efcbde*,

$E-i_sR_s-i_xR_x=0$ ...(3)

Applying the loop theorem to the path *cabc*,

$i_1+R+ir-i_sR_s=0$ ...(4)

Applying the loop theorem to the path *abda*,

$ir+i_xR_x-i_2R=0$ ...(5)

Solving (1), (2), (4) and (5),

$i_s=\dfrac{i\,(2\,r+R+R_x)}{R_s+R_x}$ ...(6)

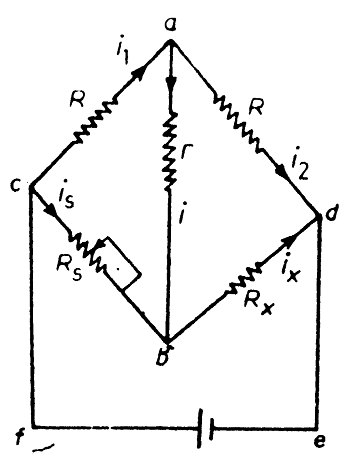


Fig. 32.16

Eluminating i_s between (1) and (3)

$$E=(R_s+R_x)\,i_s+R_s i \qquad \ldots(7)$$

Eliminating i_s between (6) and (7), and re-arranging

$$i=\frac{E(R_s-R_x)}{(R+2\,r)\,(R_s+R_x)+2R_sR_x}$$

The current $i=0$ if $R_s=x$. Already, $R_1=R_2$. This result is therefore consistent with the result of Problem 32.15 *viz.,*

$$R_x=R_s\,\frac{R_2}{R_1}$$

32.17. Let the resistances be R_1 and R_2.

For the series arrangement the equivalent resistance is

$$R=R_1+R_2$$

Joule heating is

$$P_1=\frac{E^2}{R}=\frac{E^2}{R_1+R_2} \qquad \ldots(1)$$

For the parallel arrangement, the equivalent resistance is

$$R=\frac{R_1R_2}{R_1+R_2}$$

Joule heating is

$$P_2=\frac{E^2}{R}=\frac{E^2\,(R_1+R_2)}{R_1R_2} \qquad \ldots(2)$$

By Problem, $P_2=5P_1$

$$\therefore \quad \frac{E^2(R_1+R_2)}{R_1R_2}=5\,\frac{E^2}{R_1+R_2}$$

$$(R_1+R_2)^2-5\,R_1R_2=0$$

$$R_2^2-3R_2R_1+R_1^2=0$$

Put $R_1=100$ ohm. We then have

$$R_2^2-300\,R_2+10^4=0$$

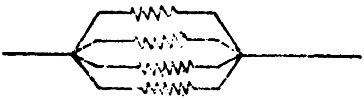
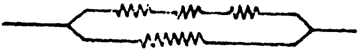
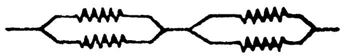
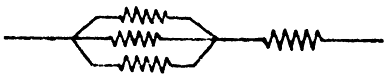
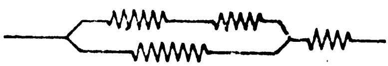
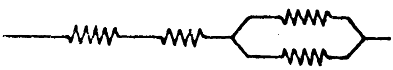
The solutions are $R_2=38$ ohm or 262 ohm.

32.18. $P=V^2/R$,

Therefore, $R_1=V^2/P=(100 \text{ volt})^2/(100 \text{ watt})$

$=100$ ohm.

Arrangement ($R_1=R_2=R_3=R_4=100$ ohm)	Equivalent resistance R	$P=\frac{E^2}{R}$ ($E=100$ volt)

(i)		25 ohm	400 watt
(ii)		75 ohm	133 watt
(iii)	(a)  (b) 	100 ohm	100 watt
(iv)		133 ohm	75 watt
(v)		167 ohm	60 watt
(vi)		250 ohm	40 watt
(vii)		400 ohm	25 watt

32.19. (*a*) Let a current i be sent at x. As the resistances in the arms xa and xb are equal, the current will divide equally, i_2 in each arm. Also the potential difference between xa and xb will also be equal so that no current flows through ab. A current equal to $i/2$ flows through ay and $i/2$ through by. The outgoing current at y is therefore again i. The potential difference between x and y is

$$V_{xy}=iR$$

where R is the equivalent resistance of the network. But

$$V_{xy}=V_{xa}+V_{ay}=(i/2)\times 10+(i/2)\times 10=10i$$

$\therefore$ $iR=10\,i$, or $R=10$ ohm.

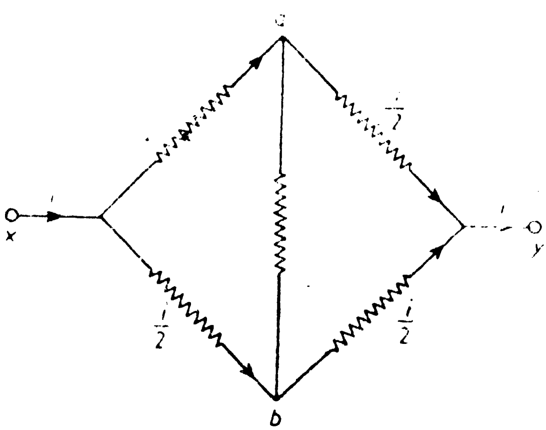


Fig. 32.19 (a)

(*b*) Let a potential difference V_{xy} exist, across the points x and y. Let a current i enter at x and a current i leave at y. If R is the equivalent resistance of the net work, then

$$iR=V_{xy} \qquad \ldots(1)$$

$$V_{xy}=V_{xA}+V_{Ay} \text{ Also,}$$

$$=10\,(i-i_1)+20\,(i-i_1-i_2)$$

or $$V_{xy}=30\,i-30i_1-20i_2 \qquad \ldots(2)$$

The potential difference between points x and B is

$$V_{xB}=V_{xC}+V_{CB}=10i_1+10i_1=20i_1 \qquad \ldots(3)$$

$$V_{xB}=V_{xA}+V_{AB}=10\,(i-i_1)+10i_2 \qquad \ldots(4)$$

Combining (3) and (4),

$$3i_1-i_2=i \qquad \ldots(5)$$

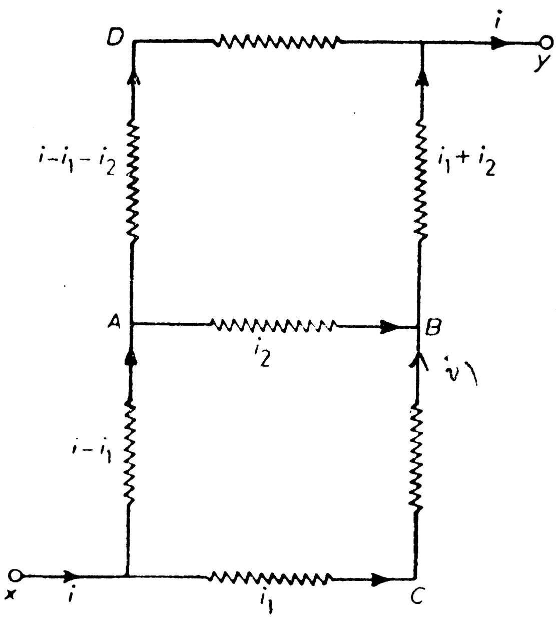


Fig. 32 19 (b)

The potential difference between A and y is

$$V_{Ay}=V_{AB}+V_{By}$$

or $$V_{Ay}=10i_2+10(i_1+i_2) \qquad \ldots(6)$$

Also, $$V_{Ay}=V_{AD}+V_{Dy}$$

$$=10(i-i_1-i_2)+10(i-i_1-i_2)$$

or $$V_{Ay}=20\,(i-i_1-i_2) \qquad \ldots(7)$$

Combining (6) and (7),

$$3i_1+4i_2=2i \qquad \ldots(8)$$

Solving (5) and (8),

$$i_1=\frac{2i}{5} \qquad ...(9)$$

$$i_2=\frac{i}{5} \qquad ...(10)$$

Using (9) and (10) in (2) and combining with (1),

$$V_{xy}=14i=iR$$

whence, $R=14$ ohms.

(*c*) Let the potential difference between the points x and y be V_{xy}. Let a current i enter at x and the same current i flow out from y.

$$V_{xy}=iR \qquad ...(11)$$

where R is the equivalent resistance of the network.

$$V_{xy}=V_{xa}+V_{a}$$

$$=10i_1+10(i_1-i_2)$$

or $$V_{xy}=20i_1-10i_2 \qquad (12)$$

Also, $$V_{xb}=V_{xa}+V_{ab}$$

$$10(i-i_1)=10i_1+10i_2$$

or $$2i_1+i_2=i \qquad ...(13)$$

Also, $$V_{ay}=V_{ab}+V_{by}$$

$$10(i_1-i_2)=10i_2+10(i-i_1+i_2)$$

or $$2i_1-3i_2=i \qquad ...(14)$$

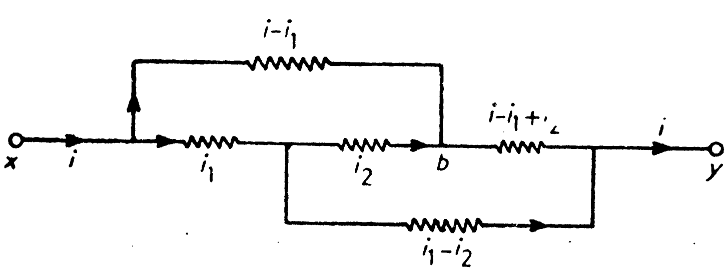


Fig 32.19 (c)

Solving (13) and (14)

$$i_1=\frac{i}{2}$$

$$i_2=0 \qquad ...(15)$$

Using (15) in (12) and combining with (11),

$$V_{xy}=iR=20i_1=10i$$

$\therefore\quad R=10$ ohms.

32.20. Applying the loop theorem to the lower loop

$$E_2-i_1R_1=0$$

or $$i_1=\frac{E_2}{R_1}=\frac{5\text{ volt}}{100\text{ ohms}}$$
$$=0.05\text{ amp}$$

Applying the loop theorem to the upper branch.

$$E_3+E_2-E_1-iR_2=0$$

$$\therefore\; i=\frac{(4+5-6)\text{ volt}}{50\text{ ohm}}$$
$$=0.06\text{ amps}$$

$$V_{ab}=E_2+E_3=(5+4)\text{ volt}$$
$$=9\text{ volt.}$$

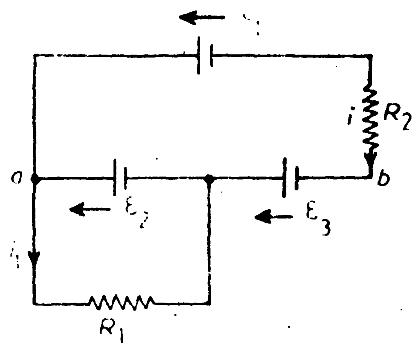


Fig. 32.20

32.21. For the series connection

$$E-iR-ir+E-ir=0$$

$$\therefore\quad i=\frac{2E}{2r+R} \qquad ...(1)$$

For the parallel connection, the internal resistances of the two batteries are in parallel, their equivalent resistance being $r/2$. For the current i through R the effective emf is E

$$\therefore\quad i=\frac{E}{R+r/2}=\frac{2E}{2R+r} \qquad ...(2)$$

The ratio of current through R for the series connection to the current for the parallel connection is obtained by diving (1) by (2),

$$\frac{i_s}{i_p}=\frac{2R+r}{R+2r}=\frac{R+(R+r)}{r+(R+r)} \qquad ...(3)$$

(*a*) $R>r$, For . (3) shows $i_s>i_p$

(*b*) $R<r$, For Eq. (3) gives $i_s<i_p$.

32.22. Applying Kirchoffs's rules to the circuit of Fig. 32.22,

$$i_1=i_2+i_3 \qquad ...(1)$$

$$E_1-i_3R_3-i_1R_1=0 \qquad ...(2)$$

$$E_2-i_2R_2-i_1R_1=0 \qquad ...(3)$$

Using the numerical values in (2) and (3) and solving the above three equations we find

$$i_1=\frac{5}{19}\text{ amp.}$$

$$i_2=\frac{3}{19}\text{ amp.}$$

$$i_4=\frac{8}{19}\text{ amp.}$$

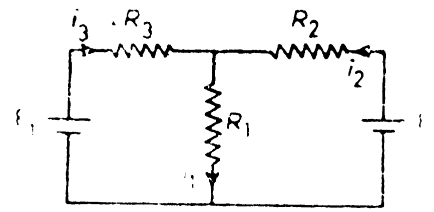


Fig 32.22

(*a*) The Joule heat produced in R_1 is

$$P_1=i_1^2\ R_1=\left(\frac{5}{19}\text{ amp }\right)^2(5\text{ ohm})=0.34626\text{ watts;}$$

that produced in R_2 is

$$P_2=i_2^2\ R_2=\left(-\frac{3}{19}\text{ amp }\right)^2(2\text{ ohm})=0.04986\text{ watts}$$

that produced in R_3 is

$$P_3=i_3^2\ R_3=\left(\frac{8}{19}\text{ amp }\right)^2(4\text{ ohm})=0.70914\text{ watts}$$

(*b*) Power supplied by, E_1 is

$$E_1i_3=(3\text{ volt})\left(\frac{8}{19}\text{ amp }\right)=1.25316\text{ watts}$$

Power supplied by E_2 is $E_2i_2=(1.0\text{ volt})\left(-\frac{3}{19}\right)=-0.15789$ watts

(*c*) Power supplied by E_1 and E_2 is $E_1i_3+E_1i_2$

$$=(1.26316-0.15789)\text{ watts}=1.10527\text{ watts.}$$

'oule heating in the three resistances is,

$$P=P_1+P_2+P_3=(9.34626+0.04986+0.70914)\text{ watts}$$

$$=1.10526\text{ watts.}$$

Thus, the Joule heating is equal to the power supplied by E_1 and E_2. The battery E_2 is charged and negligible Joule heat appears in the battery.

32.23. (*a*) The resistance R_1 is put in the middle of the range. Rough adjustment of current is made with R_2 (lower resistance) and fine adjustment is made with R_1 (higher resistance).

(*b*) The equivalent resistances of R_1 and R_2 in parallel is

$$R=\frac{R_1R_2}{R_1+R_2} \qquad \ldots(1)$$

Holding R_2 constant, differentiate (1). Change in R is

$$\Delta R=\frac{R_2^2\ \Delta R_1}{(R_1+R_2)^2}=\left(\frac{R_2}{R_1+R_2}\right)\left(\frac{R_1R_2}{R_1+R_2}\right)\frac{\Delta R_1}{R_1}$$

$$\frac{\Delta R/R}{\Delta R_1/R_1}=\frac{R_2}{R_1+R_2}$$

Setting $R_1=20R_2$, we get $\dfrac{\Delta R/R}{\Delta R_1/R_1}=\dfrac{1}{21}$

Thus, a small change in the resistance of the parallel combination is crossed by a large fractional change in R_1, thereby permitting fine adjustment.

32.24. The resistance R_V of the voltmeter is in parallel to the resistance R Fig. 32.23 (*a*) so that the effective resistance R' is given by

$$\frac{1}{R'}=\frac{1}{R}+\frac{1}{R_V}$$

or $$\frac{1}{R}=\frac{1}{R'}-\frac{1}{R_V}$$

32.25. The resistance R_A of the ammeter is in series with R, so that the effective resistance R' is given by

$$R'=R+R_A$$

or $$R=R'-R_A$$

32.26. Without ammeter resistance the current

$$i=\frac{E}{R_1+R_2+r}$$

$$=\frac{5 \text{ volt}}{(5+4+2) \text{ ohm}}=0.4545 \text{ amp}$$

With the inclusion of ammeter resistance the current is

$$i'=\frac{E}{R_1+R_2+r+R_A}=\frac{5 \text{ volt}}{(5+4+2+0.1) \text{ ohm}}$$
$$=0.4504 \text{ amp.}$$

Error in current measurement is, $\Delta i=i-i'$

Fractional error in current measurement, is

$$\frac{\Delta i}{i}=\frac{(0.4545-0.4504) \text{ amp}}{0.4545 \text{ amp}}=0.009.$$

∴ Percentage error is $100(\Delta i/i)=0.009\times100=0.9\%$

32.27. If no current goes through the voltmeter ($R_V=\infty$), the current

$$i=\frac{E}{R_1+R_2+r}=\frac{5 \text{ volt}}{(50+40-20) \text{ ohm}}$$
$$=45.45\times10^{-3} \text{ amp.}$$

Potential difference accros R_1 is

$$V=iR_1=(45.45\times10^{-3} \text{ amp})(50 \text{ ohm})=2.27 \text{ volt}$$

With a finite resistance R_V for the voltmeter, the resistance of parallel combination of R_1 and R_V is

$$R=\frac{R_1R_V}{R_1+R_V}=\frac{(50 \text{ ohm})(1000 \text{ ohm})}{(50 \text{ ohm}+1000 \text{ ohm})}=47.6 \text{ ohm.}$$

$$\text{Current, } i'=\frac{E}{R_2+r+R}=\frac{5 \text{ volt}}{(40+20+47.6) \text{ ohm}}=46.46\times10^{-3} \text{ amp}$$

Potential difference across R_1 is

$$V'=i'R=(46.46\times10^{-3} \text{ amp})(47.6 \text{ ohm})=2.21 \text{ volt.}$$

Fractional error in potential difference across R_1 is

$$\frac{\Delta V}{V}=\frac{V-V'}{V}=\frac{(2.27-2.21) \text{ volt}}{2.27 \text{ volt}}=0.026$$

∴ Percentage error in reading the potential difference is

$0.026\times100=2.6\%$

32.28. (*a*) The currents are shown

Applying junction theorem at *a*,

$$i_2+i_3=i_1 \qquad \text{...(1)}$$

Applying the loop theorem to the left loop,

$$E_2-i_2R_2-i_1R_1-E_1-i_1R_1=0 \qquad \text{...(2)}$$

Applying the loop theorem to the right loop,

$$E_3-i_3R_1+i_2R_2-E_2-i_3R_1=0 \qquad \text{...(3)}$$

Putting the numerical values, (2) and (3) become

$$i_1+i_2=1 \qquad \ldots(2)'$$

$$i_2-i_3=0 \qquad \ldots(3)'$$

Solving (1), (2)' and (3)', we get

$i_1=0.67$ amp, counter-clockwise in the left loop

$i_2=0.33$ amp, up in tbe center branch

$i_3=0.33$ amp, counter-clockwise in the right loop.

(*b*) The potential difference between *a* and *b* is

$$V_{ab}=E_2-i_2R_2=(4\text{ volt})-(0.33\text{ amp})(2\text{ ohm})=3.3\text{ volts}.$$

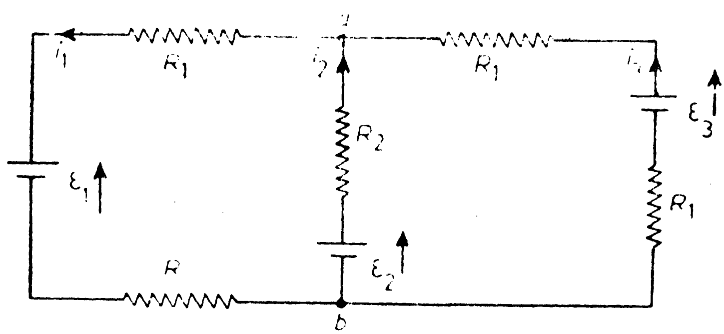


Fig 32.28

32.29. For the *RC* circuit the charge *q* after time *t* is

$$q=CE\,(1-e^{-t/RC})$$

where *CE* is the equilibrium charge on the capacitor and the product *RC* is the time constant.

$$\frac{q}{CE}=\frac{100-1}{100}=1-\frac{1}{100}=1\mp e^{-t/RC}$$

$$\therefore\ e^{-t/RC}=\frac{1}{100}$$

$$e^{t/RC}=100$$

$$t/RC=\ln 100=4.6$$

Thus, time $t=4.6$ times time constant must elapse before a capacitor in an *RC* circuit is charged to within 1.0 per cent of its equilibrium charge.

32.30

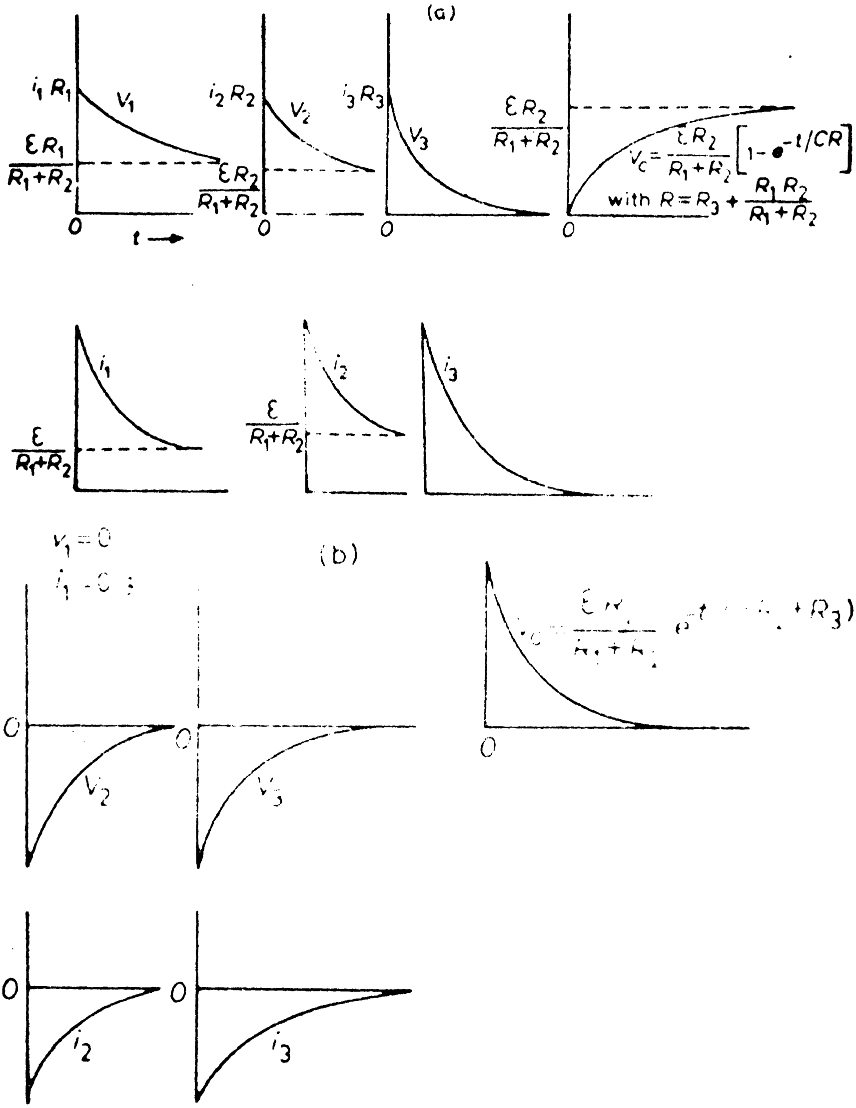


Fig. 32.30

32.31. Units of $RC = (\text{ohm})(\text{farad})$

$$= \frac{\text{volt}}{\text{ampere}} \cdot \frac{\text{coulomb}}{\text{volt}} = \frac{\text{coulomb}}{\text{ampere}}$$

$$= \frac{\text{coulomb}}{\text{coulomb/time}} = \text{time}.$$

32.32. Energy of the capacitor is

$$U = \frac{1}{2}\frac{Q^2}{C}.$$

Rate of energy transfer is

$$\frac{dU}{dt}=\frac{Q}{C}\;\frac{dQ}{dt}=\frac{Q}{C}i=Vi.$$

$$\therefore \qquad \frac{dU}{dt}=i^2R$$

The right hand side is nothing but Joule heating.

Thus, at any instant the rate of energy transfer is completely accounted for by Joule heating.

32.33. $RC=(3\times10^6\text{ ohm})(1.0\times10^{-6}\text{ f})=3$ sec.

(*a*) $$Q=Q_0\,(1-e^{-t/RC})$$

$$\therefore \qquad i=\frac{dQ}{dt}=\frac{Q_0}{RC}e^{-t/RC}=\frac{E}{R}e^{-t/RC}$$

$$=\frac{(4\text{ volt})}{(3\times10^0\text{ ohm})}e^{-(1.0\text{ sec}/3\text{ sec})}$$

$$=9.6\times10^{-7}\text{ coul/sec,}$$

(*b*) Energy $U=\dfrac{1}{2}\;\dfrac{Q^2}{C}$

$$\therefore \qquad \frac{dU}{dt}=\frac{Q}{C}\;\frac{dQ}{dt}=i\;\frac{Q}{C}=\frac{iQ_0}{C}(1-e^{-t/RC})$$

$$=iE\,(1-e^{-t/RC})=(9.6\times10^{-7}\text{ amp})(4\text{ volt})(1-e^{-0.33})$$

$$=1.08\times10^{-6}\text{ watts.}$$

(*c*) Joule heating in the resistor is

$$i^2R=(9.6\times10^{-7}\text{ amp})^2(3\times10^6\text{ ohm})=2.76\times10^{-6}\text{ watt}$$

(*d*) Energy delivered by the seat of emf is

$$Ei=(4\text{ volt})(9.6\times10^{-7}\text{ amp})=3.84\times10^{-6}\text{ watts.}$$

SUPPLEMENTARY PROBLEMS

S.32.1. The current in the circuit is

$$i=\frac{2E}{R+r_1+r_2} \qquad \ldots(1)$$

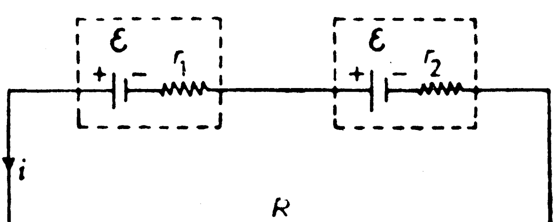


Fig. S.32.1

The potential difference across the first-battery is

$$V_1 = ir_1 = \frac{2E\,r_1}{R+r_1+r_2} \qquad \ldots(2)$$

where use has been made of (1).

By Problem, $V_1 = E$ ...(3)

Using (3) in (2),

$$\frac{2E\,r_1}{R+r_1+r_2} = E$$

whence, $R = r_1 - r_2$

S.32.2. (*a*) $P = iV$

where V is the potential difference between A and B

$$\therefore \qquad V = \frac{P}{i} = \frac{50 \text{ watts}}{1.0 \text{ amp}} = 50 \text{ volt.}$$

(*b*) The potential drop across R is

$$V_{(R)} = iR = (1.0 \text{ amp})(2.0 \text{ ohm}) = 2 \text{ volt}$$

In the absence of the internal resistance, the emf of C is 50-2 volt=48 volt.

(*c*) As the element C is opposing the current i, B is its negative terminal.

S.32.3. Power, $P_{(r)} = \frac{V^2}{r}$; $P_{(R)} = \frac{V^2}{R}$

As for parallel connection V is the same for bulbs of resistance r and R; the power P will be larger for r $(r < R)$. Hence the bulb with resistance, r will be brighter.

(*b*) Power, $P_{(r)} = i^2 r$; $P_{(R)} = i^2 R$

As for series connection i is the same for bulbs r and R; $P_{(R)}$ will be larger than $P_{(r)}$. Therefore, the bulb with R will be brighter than that with r.

S.32.4. For series connection, total resistance of the circuit is $R+Nr$. Total emf will be NE.

$$\text{Current, } i = \frac{NE}{R+Nr} \qquad \ldots(1)$$

For parallel connection, the total internal resistance will be r/N, which is in series with R. Total resistance of the circuit will then be $R+(r/N)$. The net emf will be simply E. The current will then be

$$i = \frac{E}{R+r/N} = \frac{NE}{NR+r} \qquad \ldots(2)$$

By Problem, in both (1) and (2), i is identical.

$$\text{Hence,} \qquad \frac{NE}{R+Nr} = \frac{NE}{NR+r}$$

whence $(N-1)(R-r)=0$

Since $\quad N\neq 1$, we have $R=r$.

S.32.5. (*a*) Let a current $12i$ enter at the corner A and emerge at B. Because of the arrangement with identical resistors the current

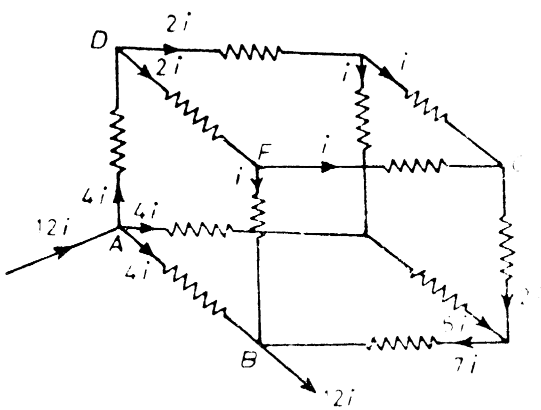


Fig. S.32.5 (a)

is divided symmetrically as shown in Fig. S.32.5 (*a*).

$$R_{AB}=\frac{V_{AB}}{12i}=\frac{V_{AD}+V_{DF}+V_{FB}}{12\,i}$$

$$=\frac{4iR+2\,iR+iR}{12\,i}=\frac{7}{12}R$$

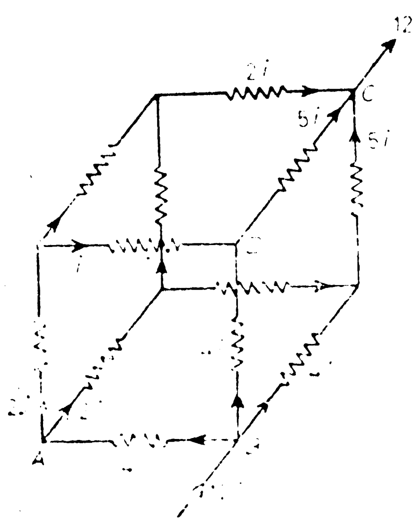


Fig. S.32.5 (b)

(*b*) Let a current 12 i enter the corner B and $12i$ emerge at C. Because of the arrangement with identical, resistors, the current is divided symmetrically in various branches as shown in Fig. S.32.5 (*b*).

$$R_{BC}=\frac{V_{BC}}{12i}=\frac{V_{BD}+V_{DC}}{12i}=\frac{4iR+5iR}{12i}=\frac{3}{4}R.$$

(*c*)

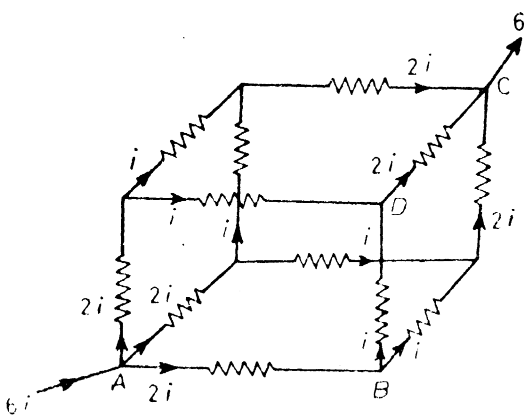


Fig. S.32.5 (c)

Let a current $6i$ enter at A and $6i$ leave at C. Because of equality of resistors, the current in various branches is divided symmetrically as shown in Fig. S.32.5 (*c*).

$$R_{AC}=\frac{V_{AC}}{6i}=\frac{V_{AB}+V_{BD}+V_{DC}}{6i}$$

$$=\frac{2iR+iR+2iR}{6i}=\frac{5}{6}R$$

S.32.6. Join one end of each of 1 ohm resistors together and convert the other ends separately to N terminals as shown in Fig. S.32.6.

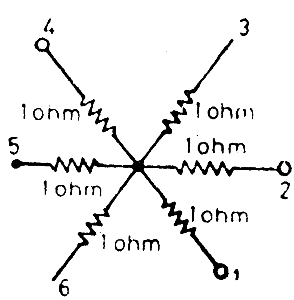


Fig. S.32.6

S.32.7. (*a*) The resistance of ammeter and R_1 are in series.

$$(R_1+3.62)\ (0.317 \text{ amp})=28.1 \text{ volt}$$

$\therefore \quad R_1=85$ ohm

(*b*) The resistance of voltmeter and R_2 are in parallel. The equivalent resistance of this combination is

$$R=\frac{307\ R_2}{307+R_2}$$

The voltage drop across R_2 is

$$iR=\frac{307\ R_2 i}{307+R_2}=V$$

$$\frac{(307\ R_2)(0.356 \text{ amp})}{307+R_2}=23.7 \text{ volt}$$

$$R_2=85 \text{ ohm.}$$

S.32.8. (*b*) For an RC circuit, the potential difference across the capacitor after time t is

$$V=E\ e^{-t/RC} \qquad \ldots(1)$$

where E is the initial potential difference and $\tau=RC$, is the time constant. Substituting the given values,

$$1.0 \text{ volt}=(100 \text{ volt})\ e^{-10/RC} \qquad \ldots(2)$$

$\therefore \quad e^{10/RC}=100/1.0=100$

Taking logarithms on both sides,

$$\frac{10}{RC}=\ln 100=4.6$$

$$\therefore \qquad \tau=RC=\frac{10 \text{ sec}}{4.6}=2.17 \text{ sec.}$$

(*a*) After 20 secs, the potential difference would be,

$$V=(100 \text{ volt})\ e^{-20 \text{ sec}/2.17 \text{ sec}}=0.01 \text{ volt.}$$

S.32.9. (*a*) $U_0=\dfrac{q_0^2}{2C}$

$$\therefore \qquad q_0=\sqrt{2U_0C}=\sqrt{(2)(0.5 \text{ joule})(10^{-6} \text{ farad})}$$
$$=10^{-3} \text{ coulombs.}$$

(*b*) $\qquad q=q_0 e^{-t/RC}$

$$\therefore \qquad i=\frac{dq}{dt}=-\frac{q_0}{RC}e^{-t/RC}$$

At $\quad t=0,\ i=\dfrac{10^{-3}\text{ coul}}{(10^{6}\text{ ohm})\,(10^{-6}\text{ farad})}=10^{-3}\text{ amp}$

(*c*) $\quad V_C=\dfrac{q}{C}=\dfrac{q_0e^{-t/RC}}{C}=\dfrac{10^{-3}\text{ coul}}{10^{-6}\text{ farad}}\ e^{-t}=1000\,e^{-t}\text{volt}$

Since $RC=(10^{6}\text{ ohms})(10^{-6}\text{ f})=1\text{ sec}$

$$V_R=Ri=R\left(-\frac{q_0}{RC}\right)e^{-t/RC}=100e^{-t}\text{ volt}$$

(*d*) Rate of Joule heating, $U_J=i^2R=\left(\dfrac{q_0}{RC}\right)^2Re^{-2t/RC}$

$$=\left[\frac{(10^{-3}\text{ coul})}{(10^{6}\text{ ohm})\,(10^{-6}\text{ farad})}\right]^2(10^{6}\text{ ohm})\,e^{-2t}$$

$$=e^{-2t}\text{ watt}$$

S.32.10. (*a*) At $t=0$, C is to be considered closed.

Applying junction theorem at the junction of R_1 and R_2,

$$i_1=i_2+i_3 \qquad \ldots(1)$$

Applying Kirchoff's law to the lower loop,

$$E-i_1R_1-i_2R_2=0 \qquad \ldots(2)$$

Applying Kirchoff's law to the upper loop,

$$i_2R_2-i_3R_3=0 \qquad \ldots(3)$$

Since $\quad R_2=R_3$,

$$i_3=i_2 \qquad \ldots(4)$$

Using (4) in (1),

$$i_2=\tfrac{1}{2}i_1 \qquad \ldots(5)$$

Using (5) in (2) and the fact that $R_1=R_2$,

$$\frac{3}{2}R_1i_1=E$$

or $\quad i_1=\dfrac{2}{3}\dfrac{E}{R_1}=\dfrac{2}{3}\dfrac{(1200\text{ volt})}{(7.3\times10^{5}\text{ ohm})}=1.1\times10^{-3}\text{ amp}$

$$i_3=i_2=\tfrac{1}{2}i_1=\tfrac{1}{2}(1.1\times10^{-3}\text{ amp})=0.55\times10^{-3}\text{ amp.}$$

At $t=\infty$, the capacitor C is fully charged and C must be considered as open. In that case $i_3=0$ and $i_1=i_2$.

From (2), $i_1R_1+i_2R_2=E$

Since $R_2=R_1$,

$$i_1=i_2=\frac{E}{2R_1}=\frac{1200\text{ volt}}{2\times7.3\times10^{5}\text{ ohm}}=0.82\times10^{-3}\text{ amp}$$

$$i_3=0.$$

(*b*)

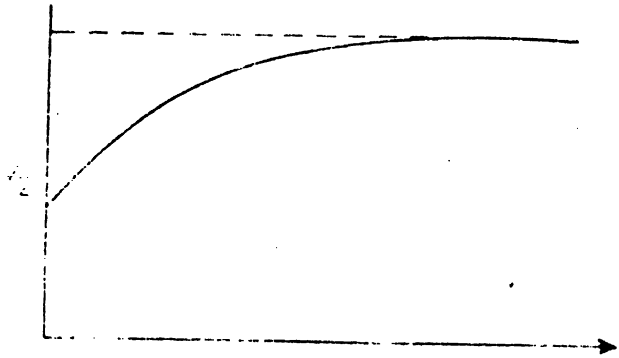


Fig. S.32.10

(*c*) At $t=0$, $i_2=0.55\times 10^{-3}$ amp

$\therefore$ $V_2=i_2R_2=(0.55\times 10^{-3}\text{ amp})(7.3\times 10^5\text{ ohms})=401$ volt

At $t=\infty$, $i_2=0.82\times 10^{-3}$ amp

$\therefore$ $V_2=i_2R_2=(0.82\times 10^{-3}\text{ amp})(7.3\times 10^5\text{ ohm})=599$ volt.

(*d*) The voltage drop across R_2 is seen to approach its final value asymtotically, and hence an infinite time is required for the voltage drop to develop to its maximum value. However, the time for the voltage to increase to any stated fraction of its final value is quite definite, and for the usual values of R and C encountered in common practice the voltage grows essentially to its final value within a reasonably short time. Let time be measured in terms of time constant $\tau=RC$.

$$V_2=V_\infty(1-e^{-t/R_2C}) \qquad \ldots(6)$$

$$\tau=R_2C=(7.3\times 10^5\text{ ohm})(6.5\times 10^{-6}\text{ farad})=4.75\text{ secs}$$

Set $t=\tau$ in (6) to find

$$V_2/V_\infty=1-e^{-1}=0.63$$

Next set $t=2\tau$ in (6) to find

$$\frac{V_2}{V_\infty}=1-e^{-2}=0.87$$

Thus, for the fraction $V_2/V_\infty=0.63$, we have $t=\tau=R_2C=4.75$ secs, and for the fraction $V_2/V_\infty=0.87$ we have $t=2\tau=2R_2C=9.5$ sec.

33 THE MAGNETIC FIELD

33.1. (*a*) The beam will deflect to the east. The direction of deflection is found from the rule for the vector product in the relation

$$\mathbf{F}=q\mathbf{v}\times\mathbf{B}.$$

(*b*) The velocity of electron whose kinetic energy is K can be found from

$$v=\sqrt{\frac{2K}{m}}=\sqrt{\frac{2\times(12\times10^{3}\text{ eV})(1.6\times10^{-19}\text{ joule/eV}}{9.1\times10^{-31}\text{ kg}}}$$

$$=6.5\times10^{7}\text{ meter/sec}$$

Force on electron,

$$F=q\mathbf{v}\times\mathbf{B}=qvB\sin\theta=qvB$$

since, $\theta=90°$.

$$F=(1.6\times10^{-19}\text{ coul})(6.5\times10^{7}\text{ meter/sec})(5.5\times10^{-5}\text{ weber/meter}^2).$$

$$=5.7\times10^{-16}\text{ nt}$$

$\therefore$ Acceleration, $a=\dfrac{F}{m}=\dfrac{5.7\times10^{-16}\text{ nt}}{9.1\times10^{-31}\text{ kg}}=6.3\times10^{14}\text{ meter/sec}^2.$

(*c*) Time taken to traverse 0.2 meter of horizontal distance along south-north direction,

$$t=\frac{s}{v}=\frac{0.2\text{ meter}}{6.5\times10^{7}\text{ meter/sec}}=3.1\times10^{-9}\text{ sec}$$

Deflection in the eastern direction is given by

$$y=\tfrac{1}{2}at^2$$

$$=\tfrac{1}{2}(6.3\times10^{14}\text{ meter/sec}^2)(3.1\times10^{-9}\text{ sec})^2$$

$$=3\times10^{-3}\text{ meter}$$

$$=3.0\text{ mm}.$$

33.2. The kinetic energy of a particle of mass m and charge q, moving in a circular orbit of radius R under the influence of magnetic field B, is

$$K=\frac{q^2B^2R^2}{2m}$$

(*a*) For α-particle, $q=2e$ and $m_\alpha=4m_p$,

$$\frac{K_\alpha}{K_p}=\left(\frac{q_\alpha}{q_p}\right)^2\left(\frac{m_p}{m_\alpha}\right)$$
$$=(2)^2(1/4)=1$$

$\therefore \quad K_\alpha=K_p=1$ Mev

(*b*) For duetron, $q=e$ and $m_d=2m_p$,

$$\frac{K_d}{K_p}=\left(\frac{q_d}{q_p}\right)^2\left(\frac{m_p}{m_d}\right)=(1.0)^2\left(\frac{1}{2}\right)=\frac{1}{2}$$

$\therefore \quad K_d=\frac{1}{2}K_p=\frac{1}{2}$Mev

33.3. The magnitude of force on wire of length l, carrying current i in a uniform magnetic field B at 30° to the wire, is

$$F=ilB\sin\theta$$
$$=(10\text{ amp})(1.0\text{ meter})(1.5\text{ webers/meter}^2)(\sin 30°)$$
$$=7.5\text{ nt.}$$

The force acts perpendicular to the wire and the magnetic field.

33.4. Magnetic force on the wire is

$$F=i\mathbf{l}\times\mathbf{B}=ilB$$

since B is at right angles to the displacement vector **l**.

Setting, $F=mg$, the weight of the wire, the magnitude of the current required to remove the tension in the supporting leads is

$$i=\frac{mg}{lB}=\frac{(1.0\times10^{-2}\text{ kg})(9.8\text{ meter/sec}^2)}{(0.6\text{ meter})(0.4\text{ weber/meter}^2)}$$

$$=0.41\text{ ampere}$$

in the direction from left to right.

33.5. Magnetic force, $F=Bqv$.

The dimensional formula for magnetic induction is

$$[B]=\frac{[F]}{[q]\,[v]}=\frac{[MLT^{-2}]}{[Q]\,[LT^{-1}]}=[MT^{-1}Q^{-1}].$$

The flux ϕ_B for a magnetic field is defined by

$$\phi_B=\int \mathbf{B}.d\mathbf{S}$$

Thus, the dimensions of ϕ_B are equal to that of B multiplied by that of area. That is,

$$[\phi_B]=[B]\,[S^2]=[MT^{-1}Q^{-1}][L^2]=[ML^2T^{-1}Q^{-1}]$$

33.6. The force on the wire is

$$F = ilB$$

Acceleration, $a = \dfrac{F}{m} = \dfrac{ilB}{m}$

Velocity, $v = v_0 + at = 0 + at = \dfrac{ilBt}{m}$

directed away from the generator.

33.7. The force acting on the wire is

$$F = Bil \quad \ldots(1)$$

$$mv = \int F dt = \int Bil\, dt = Bl \int i dt = Blq \quad \ldots(2)$$

where use has been made of (1).

Also, $v = \sqrt{2gh}$

$$\therefore \quad q = \frac{m\sqrt{2gh}}{Bl} = \frac{(1.0 \times 10^{-2}\text{ kg})\sqrt{(2)(9.8\text{ meter/sec}^2)(3.0\text{ meter})}}{(0.1\text{ weber/meter}^2)(0.2\text{ meter})}$$

$$= 3.8\text{ coul.}$$

33.8. Resolve B into two mutually perpendicular components (*i*) horizontal component, $B \sin \theta$, radially outward in the plane of paper, (*ii*) vertical component, $B \cos \theta$, perpendicular to the ring out of the paper. Consider a pair of points, diametrically opposite on the ring. While the vertical component is the same in direction as well as magnitude, the current has reversed its direction, resulting in the cancellation of force. Adding up the contributions of symmetrically situated pairs of points on the ring, the force on the ring due to vertical component vanishes altogether. On the other hand, the direction of the radially outward component having fixed orientation of 90° to the direction of current at each point on the ring alone makes the contribution to the force.

$$F = (\text{current})(\text{circumference})(\text{radial component of } B)$$

$$= (i)(2\pi a)(B \sin \theta) = 2\pi aiB \sin \theta.$$

33.9. Magnetic force on the wire is

$$F = iBl$$

Setting the magnetic force equal to the frictional force,

$$F = iBl = \mu mg$$

$$\therefore \quad B = \frac{\mu mg}{il} = \frac{(0.6)(0.3 \times 0.4536\text{ kg})(9.8\text{ meter/sec}^2)}{(50\text{ amp})(0.3048\text{ meter})}$$

$$= 0.0524\text{ weber/meter}^2$$

$$= \left(0.0524\ \frac{\text{weber}}{\text{m}^2}\right)(10^4\text{ gauss}) = 524\text{ gauss.}$$

The direction of the field is normal to the plane of tracks.

33.10. Replace the wire by a series of steps parallel and perpendicular to the straight line joining a and b. Traversing along the steps we notice that in going from a to b the current flows as much in the upward direction as in the downward direction so that the net component of force due to the segments of wire in the direction perpendicular to ab is zero. On the other hand the current in the segments of wire parallel to ab is only in one direction and the total length effectively traversed is equal to ab along a straight line. Hence, the force on the wire is the same as that on the straight wire carrying a current i directly from a to b.

33.11. The torque on the coil is

$$\tau = NiAB \sin \theta$$

where N is the number of turns, i the current, A the area of the coil, B the magnetic induction, and θ the angle which the normal to the plane of the loop makes with the direction of B.

As the angle between the plane of the loop and the magnetic field is 30°, it follows that $\theta = 90° - 30° = 60°$.

$$\therefore \quad \tau = (20)(0.1 \text{ amp})(0.1 \times 0.05 \text{ meter}^2)(0.5 \text{ weber/meter}^2) \sin 60°$$

$$= 4.3 \times 10^{-3} \text{ nt-meter}$$

The torque vector is parallel to the 10 cm side of the coil and points down.

33.12. The torque is given by

$$\tau = NiAB \sin \theta \tag{1}$$

Maximum torque is obtained when $\theta = 90°$.

$$L = 2\pi r N \tag{2}$$

where r is radius of the circular coil and N is the number of turns.

Area of coil, $A = \pi r^2$ ...(3)

Eliminating r between (2) and (3),

$$A = \frac{L^2}{4\pi N^2} \tag{4}$$

Using (4) in (1), with $\theta = 90°$,

$$\tau = \frac{L^2 iB}{4\pi N} \tag{5}$$

τ is maximum when $N = 1$. This gives

$$\tau_{(max)} = \frac{L^2 iB}{4\pi}$$

33.13. Imagine the flat surface enclosed by the given loop to be divided by a fine grid into a large number of rectangles. If the same current i flows around the perimeters of the small rectangles clockwise then the currents in each rectangle is cancelled by the currents around the perimeters of the surrounding rectangles, except at the edges of the grid. The net result is a current i flowing clockwise in the large closed loop (Fig. 33.13). Let the magnetic induction B make an angle θ with the normal to the surface of the loop. Now the area of the large loop is equivalent to the areas of all the rectangles into which it is divided, and since the same current i flows through the small rectangles, it follows that the relation, $\tau = NAi\ B \sin\theta$, is valid for a loop of arbitrary shape where N is the number of turns.

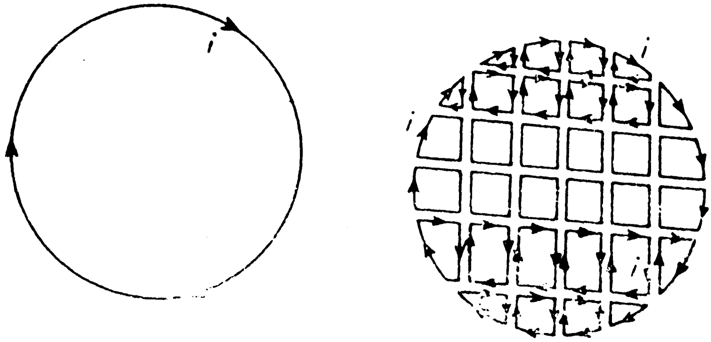


Fig. 33.13

33.14. The torque which tends to make the cylinder roll down the incline is

$$\tau = mgR \sin\theta$$

The torque acting on the wire loop by the current is

$$\tau' = N\,i\,AB \sin\theta = 2NiLRB \sin\theta$$

where we have written $2LR$ for A.

Condition that the cylinder may not slip for minimum current is

$$\tau' = \tau$$

$$2NiLRB \sin\theta = mgR \sin\theta$$

$$\therefore \quad i = \frac{mg}{2NLB} = \frac{(0.25 \text{ kg})(9.8 \text{ meter/sec}^2)}{(2)(10)(0.1 \text{ meter})(0.5 \text{ weber/meter}^2)}$$

$$= 2.45 \text{ amps.}$$

33.15. (a) Current density, $j = \dfrac{i}{A} = \dfrac{50 \text{ amp}}{(0.02 \text{ meter})(1.0 \times 10^{-3} \text{ meter})}$

$$= 25 \times 10^5 \text{ amp/meter}^2$$

If n is the number of conduction electrons/unit volume, then drift speed of the electrons,

$$v_d=\frac{j}{ne}=\frac{(25\times10^{5}\text{ amp/meter}^2)}{(1.1\times10^{29}/\text{meter}^3)(1.6\times10^{-19}\text{ coul})}$$

$$=1.4\times10^{-4}\text{ meter/sec}$$

(*b*) Magnetic force is

$$F=qv_dB=(1.6\times10^{-19}\text{ coul})(1.4\times10^{-4}\text{ meter/sec})\left(2\ \frac{\text{weber}}{\text{meter}^2}\right)$$

$$=4.5\times10^{-23}\text{ nt},$$

in the downward direction.

(*c*) Set the electric field equal to magnetic field,

$$Ee=F$$

$$\therefore\qquad E=\frac{F}{e}=\frac{4.5\times10^{-23}\text{ nt}}{1.6\times10^{-19}\text{ coul}}=2.8\times10^{-4}\text{ volt/meter},$$

in the downward direction.

(*d*) Necessary voltage to produce this field is

$$V=Eh=(2.8\times10^{-4}\text{ volt/meter})(0.02\text{ meter})$$

$$=5.6\times10^{-6}\text{ volt}.$$

Top voltage should be +ve and bottom —ve.

(*e*) $\mathbf{E}_H=-\mathbf{v}_d\times\mathbf{B}$

The magnitude of E_H is given by

$$v_dB=(1.4\times10^{-4}\text{ meter/sec})(2\text{ weber/meter}^2)$$

$$=2.8\times10^{-4}\text{ volt/meter},$$

in the downward direction.

33.16. (*a*) $E_H=\dfrac{jB}{ne}$

where j is the current density, B the magnetic induction and n the number of charge carriers per unit volume.

$$\frac{E_H}{E}=\frac{jB}{Ene}=\frac{B}{ne\rho}$$

where we have used the expression for resistivity $\rho=E/j$.

(*b*) 90°

(*c*) Textbook Example 5 gives,

$$B=1.5\text{ webers/meter}^2$$

For copper, $n=8.4\times10^{28}/\text{meter}^3$

$$\rho=1.7\times10^{-8}\text{ ohm-meter}$$

Therefore, $\frac{E_H}{E}=\frac{B}{ne\rho}$

$$=\frac{1.5 \text{ webers/meter}^2)}{(8.4\times10^{28}/\text{meter}^3)(1.6\times10^{-19}\text{ coul})(1.7\times10^{-8}\text{ ohm-meter})}$$
$=0.0066.$

33.17. (*a*) Kinetic energy of proton, $K_p=eV$

Kinetic energy of deutron, $K_d=eV$

Kinetic energy of α-particle, $K_\alpha=2eV$

$K_p : K_d : K_\alpha=1 : 1 : 2$

(*b*) Radius of curvature is

$$r=\frac{\sqrt{2mK}}{qB}=\sqrt{\frac{2mV}{qB^2}}$$

$$\frac{r_d}{r_p}=\sqrt{\frac{m_d}{m_p}\ \frac{q_p}{q_d}}$$

But $\frac{m_d}{m_p}=2$ and $\frac{q_p}{q_d}=\frac{e}{e}=1$

$\therefore$ $r_d=\sqrt{2}r_p=\sqrt{2}(10\text{ cm})=14.1\text{ cm}$

$$\frac{r_\alpha}{r_p}=\sqrt{\frac{m_\alpha}{m_p}\ \frac{q_p}{q_\alpha}}=\sqrt{\frac{4}{1}\times\frac{1}{2}}=\sqrt{2}$$

$r_\alpha=\sqrt{2}r_p=\sqrt{2}(10\text{ cm})=14.1\text{ cm}$

33.18. $R=\frac{\sqrt{2mK}}{qB}$

$$\frac{R_d}{R_p}=\sqrt{\frac{m_d}{m_p}\ \frac{q_p}{q_d}}$$

But, $m_d/m_d=2$ and $q_p/q_d=1$

$\therefore$ $R_d/R_p=\sqrt{2}$

or $R_d=\sqrt{2}\ R_p$

$$\frac{R_\alpha}{R_p}=\sqrt{\frac{m_\alpha}{m_p}}\left(\frac{q_p}{q_\alpha}\right)=\sqrt{4}(1/2)=1$$

or $R_\alpha=R_p$

33.19. (*a*) Speed, $v=\frac{qBR}{m}$

$$=\frac{(2)(1.6\times10^{-19}\text{ coul})(1.2\text{ weber/meter}^2)(0.45\text{ meter})}{(6.68\times10^{-27}\text{ kg}}$$
$=2.6\times10^7$ meter/sec.

(*b*) Time period, $T=\dfrac{2\pi m}{qB}=\dfrac{(2\pi)(6.69\times10^{-27}\text{ kg})}{(2\times1.6\times10^{-19}\text{ coul})(1.2\text{ weber/meter}^2)}$

$=1.09\times10^{-7}$ sec.

(*c*) Kinetic energy,

$$K=\tfrac{1}{2}mv^2=\tfrac{1}{2}(6.68\times10^{-27}\text{ kg})(2.6\times10^7\text{ meter/sec})^2$$

$$=22.5\times10^{-13}\text{ joules}$$

$$=(22.5\times10^{-13}\text{ joules})\left(\frac{1\text{ Mev}}{1.6\times10^{-13}\text{ oule}}\right)$$

$$=14.1\text{ Mev.}$$

(*d*) Set $K=qV=2eV$

$\therefore\quad 2eV=14.1\times10^6\ ev$

or $\quad V=\dfrac{14.1\times10^6\ ev}{2e}=7\times10^6$ volt.

33.20. Kinetic energy of electron, $K=15000$ eV

$$=(15000\text{ eV})\left(1.6\times10^{-19}\frac{\text{joule}}{\text{eV}}\right)=2.4\times10^{-15}\text{ joule}$$

Magnetic induction, $B=250$ gauss$=0.025$ weber/meter^2

Radius, $R=\dfrac{\sqrt{2mK}}{qB}=\dfrac{[(2)(9.1\times10^{-31}\text{ kg})(2.4\times10^{-15}\text{ joule})]^{1/2}}{(1.6\times10^{-19}\text{ coul})(0.025\text{ weber/meter}^2)}$

$=0.0165$ meter$=1.65$ cm.

33.21. First we shall investigate the path of an electron starting at rest from the positive plate. Choose the origin at O. The electric field E acts along the y-direction perpendicular to the plates. The positive plate is along the x-axis and magnetic field is applied along the z-axis perpendicular to the plane of paper. The electric force on the electron is directed along the y-axis and since the magnetic field B is along tne z-axis, and further if the component of initial velocity parallel to B is zero, then the path of electron will be contained entirely in the xy plane.

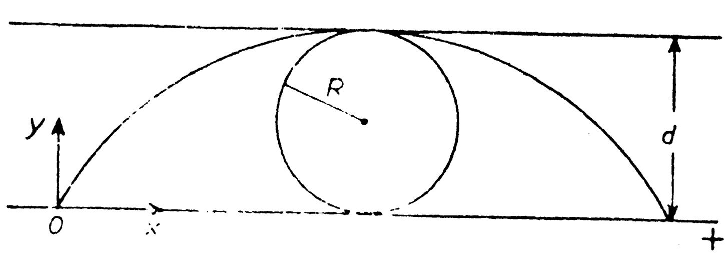


Fig. 33.21

Writing down the force equations,

$$f_y = m\frac{dv_y}{dt} = eE - eBv_x \quad \ldots(1)$$

$$f_x = m\frac{dv_x}{dt} = eBv_y \quad \ldots(2)$$

Writing for convenience,

$$\omega = \frac{eB}{m} \qquad \text{and } U = \frac{E}{B}, \quad \ldots(3)$$

Equations (1) and (2) can be re-written as

$$\frac{dv_y}{dt} = \omega U - \omega v_x \quad \ldots(4)$$

$$\frac{dv_x}{dt} = \omega v_y \quad \ldots(5)$$

Differentiating (4) and using (5),

$$\frac{d^2v_y}{dt^2} = -\omega\,\frac{dv_x}{dt} = -\omega^2 v_y$$

or
$$\frac{d^2v_y}{dt^2} + \omega^2 v_y = 0 \quad \ldots(6)$$

The solution of (6) with the initial conditions, $v_x = v_y = 0$, is seen to be

$$v_y = U \sin \omega t \quad \ldots(7)$$

Substituting (7) in (5) and solving,

$$v_x = U - U \cos \omega t \quad \ldots(8)$$

The coordinates x and y can be found out by integrating separately (7) and (8) with the initial conditions $x = y = 0$.

$$y = \frac{U}{\omega}\,(1 - \cos \omega t) \quad \ldots(9)$$

$$x = Ut - \frac{U}{\omega}\sin \omega t \quad \ldots(10)$$

Setting $\theta = \omega t$ and $R = \dfrac{U}{\omega}$, $\quad \ldots(11)$

Equations (9) and (10) respectively, become

$$y = R\,(1 - \cos \theta) \quad \ldots(12)$$

$$x = R(\theta - \sin \theta) \quad \ldots(13)$$

Equations (12) and (13) are the parametric equations of cycloid, defined as the path generated by a point on the circumference of a

circle of radius R which rolls along the x-axis. The maximum displacement of electron along the y-axis is equal to the diameter of the rolling circle i.e., $2R$.

Identifying

$$2R=d \qquad ...(14)$$

$$\frac{d}{2}=R=\frac{U}{\omega}=\frac{E/B}{eB/m}=\frac{Em}{eB^2} \qquad ...(15)$$

Also, $$E=\frac{V}{d} \qquad ...(16)$$

Using (16) in (15), gives

$$B=\sqrt{\frac{2Vm}{ed^2}}$$

Thus, the condition that no electron should strike the positive plate is

$$B>\sqrt{\frac{2Vm}{ed^2}}$$

33.22. The magnitude of the magnetic deflecting force F is given by

$$F=qvB\sin\theta \qquad ...(1)$$

where θ is the angle between v and B.

Equating the magnetic force to the centripetal force, and setting $\theta=90°$ in (1),

$$qvB=\frac{mv^2}{r}$$

or $$r=\frac{mv}{qB}=\frac{p}{qB}$$

where $p=mv$, is the momentum and r is the radius of curvature.

Thus, $r \propto p$.

33.23. Equating the magnetic force to the centripetal force,

$$qvB\sin\theta=\frac{mv^2}{R}$$

or $$B=\frac{mv}{qR\sin\theta}$$

Required magnetic field will have maximum value when θ is maximum *i.e.*, 90°. Putting $\theta=90°$ in the above equation,

$$B=\frac{(1.67\times10^{-27}\text{ kg})(1.0\times10^{7}\text{ meter/sec})}{(1.6\times10^{-19}\text{ coul})(6.4\times10^{6}\text{ meter})\sin 90^\circ}$$

$$=1.63\times10^{-8}\text{ webers/meter}^2.$$

The magnetic field must act horizontally in the direction perpendicular to the equator.

33.24. Momentum of deutron is

$$p=qBr=(1.6\times10^{-19}\text{ coul})(1.5\text{ webers/meter}^2)(2.0\text{ meter})$$

$$=4.8\times10^{-19}\text{ kg}\cdot\text{meter/sec}.$$

Kinetic energy of duetron at the time of break-up is

$$K_d=\frac{p^2}{2m_d}=\frac{(4.8\times10^{-19}\text{ kg-m/sec})^2}{(2)(3.34\times10^{-27}\text{ kg})}=3.45\times10^{-11}\text{ joule}$$

Half of this energy is carried by neutron which moves tangentially to the original path, as it is unaffected by the magnetic field. The remaining half of the energy is carried by proton. So

$$K_p=\tfrac{1}{2}K_d=1.73\times10^{-11}\text{ joule}.$$

Momentum of proton is

$$p'=\sqrt{2K_pm_p}=\sqrt{(2)(1.73\times10^{-11}\text{ joule})(1.67\times10^{-27}\text{ kg})}$$

$$=2.40\times10^{-19}\text{ kg-meter/sec}$$

Radius of the new orbit is

$$r'=\frac{p'}{qB}=\frac{2.4\times10^{-19}\text{ kg-meter/sec}}{(1.6\times10^{-19}\text{ coul})(1.5\text{ weber/meter}^2)}$$

$$=1.0\text{ meter}.$$

Thus, proton moves in a circular path of radius 1.0 meter.

33.25. Resolving $\mathbf{v}$ along B and perpendicular to it, we have

$$v_{\parallel}=v\cos\theta=\sqrt{\frac{2K}{m}}\cos\theta$$

$$v_{\perp}=v\sin\theta=\sqrt{\frac{2K}{m}}\sin\theta$$

where θ is the angle between **B** and **v**.

The non-zero component of velocity along B viz. $v_{\parallel}$ makes the plane of the circular orbit of the positron path advance along the B-axis. In other words, the actual path of the positron is helical.

$$K=2\text{keV}=(2\times10^{3}\text{ eV})(1.6\times10^{-19}\text{ joul/eV})=3.2\times10^{-16}\text{ joule}$$

$$v=\sqrt{\frac{(2)(3.2\times10^{-16}\text{ joule})}{(9.1\times10^{-31}\text{ kg})}}=2.7\times10^{7}\text{ meter/sec}$$

$$v_\perp = v \sin 89° = 2.7 \times 10^7 \frac{\text{meter}}{\text{sec}}.$$

$$v_\parallel = v \cos 89° = 4.7 \times 10^5 \frac{\text{meter}}{\text{sec}}.$$

$$\therefore\ T = \frac{2\pi m}{Bq} = \frac{(2\pi)(9.1 \times 10^{-31}\ \text{kg})}{(0.1\ \text{weber/meter}^2)(1.6 \times 10^{-19}\ \text{coul})}$$

$$= 3.6 \times 10^{-10}\ \text{sec}.$$

Radius of the helix, $r = \dfrac{mv_\perp}{Bq} = \dfrac{(9.1 \times 10^{-31}\ \text{kg})(2.7 \times 10^7\ \text{meter/sec})}{(0.1\ \text{weber/meter}^2)(1.6 \times 10^{-19}\ \text{coul})}$

$$= 1.5 \times 10^{-3}\ \text{meter}$$

$$= 1.5\ \text{mm}.$$

Pitch, $P = v_\parallel T = (4.7 \times 10^5\ \text{meter/sec})(3.6 \times 10^{-10}\ \text{sec})$

$$= 0.17 \times 10^{-3}\ \text{meter}$$

$$= 0.17\ \text{mm}.$$

33.26. (*a*) Path radius, $r = \dfrac{mv}{Bq}$

$$= \frac{(9.1 \times 10^{-31}\ \text{kg})(0.1 \times 3 \times 10^8\ \text{meter/sec})}{(0.5\ \text{weber/meter}^2)(1.6 \times 10^{-19}\ \text{coul})}$$

$$= 3.4 \times 10^{-4}\ \text{meter} = 0.34\ \text{mm}.$$

(*b*) Kinetic energy, $T = \frac{1}{2}mv^2 = \frac{1}{2}(9.1 \times 10^{-31}\ \text{kg})(0.1 \times 10^8\ \text{meter/sec})^2$

$$= 4.1 \times 10^{-16}\ \text{joule}$$

$$= \frac{(4.1 \times 10^{-16}\ \text{joule})}{(1.6 \times 10^{-19}\ \text{joule/ev})} = 2.56 \times 10^3\ \text{eV}$$

$$= 2.56\ \text{kev}.$$

33.27. Period,

$$T = \frac{1}{\nu} = \frac{2\pi m}{qB}$$

$$\therefore\quad m = \frac{TqB}{2\pi} = \frac{1}{2\pi}(1.29 \times 10^{-3}\ \text{sec}/7)(1.6 \times 10^{-19}\ \text{coul})$$

$$(4.5 \times 10^{-2}\ \text{weber/meter}^2)$$

$$= 2.11 \times 10^{-25}\ \text{kg}.$$

33.28. Kinetic energy of the ion as it is accelerated through potential difference V is

$$K = qV \qquad \ldots(1)$$

As the magnetic force is equal to the centripetal force,

$$qBv=\frac{Mv^2}{r}$$

or $$v^2=\frac{q^2B^2r^2}{M^2} \qquad \ldots(2)$$

But from (1), $K=\frac{1}{2}Mv^2=qV$

or $$v^2=\frac{2qV}{M} \qquad \ldots(3)$$

Comparing (2) and (3),

$$\frac{q^2B^2r^2}{M^2}=\frac{2qV}{M}$$

or $$M=\frac{qB^2r^2}{2V}=\frac{B^2qx^2}{8V}$$

where we have set $r=x/2$.

33.29. (*a*) $F=-qvB$, Fig. 33.29.

For clockwise motion the magnetic force will be directed towards the proton. As the electrical force will also be directed towards the proton, the net centripetal force will increase. Set $q=e$.

$$F=F_e+F_B=\frac{e^2}{4\pi\varepsilon_0 r^2}+Bev=\frac{mv^2}{r} \quad \ldots(1)$$

In the presence of magnetic field, Eq. (1) shows that v would increase, leading to an increase of the angular frequency, $\omega=vr$ (for a constant radius r).

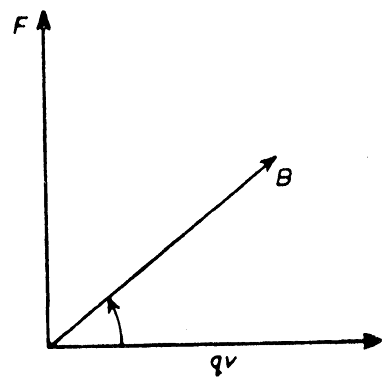


Fig. 33.29.

(*b*) For counter-clockwise motion, the centripetal force would decrease resulting in a decrease of angular frequency.

33.30. Taking into account both the possibilities of increase and decrease of ω, Eq. (1) may be re-written as

$$\frac{e^2}{4\pi\varepsilon_0 r^2}\pm Bev=\frac{mv^2}{r} \qquad \ldots(2)$$

Writing $v=\omega r$, (2) upon re-organization becomes

$$\omega^2 \pm \frac{Be}{m}\omega-\frac{e^2}{4\pi\varepsilon_0 mr^3}=0 \qquad \ldots(3)$$

Without magnetic field, (1) is simplified to

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}=\frac{mv_0^2}{r}$$

Whence, $\omega_0^2=\frac{v_0^2}{r^2}=\frac{e^2}{4\pi\varepsilon_0\, mr^3}$. ...(4)

Using (4) in (3) we have

$$\omega^2 \pm \frac{Be}{m}\omega-\omega_0^2=0$$

Solution of the above quadratic equation yields,

$$\omega=\pm\frac{Be}{2m}+\sqrt{\frac{B^2e^2}{4m^2}+\omega_0^2}$$

$$=\pm\frac{Be}{2m}+\omega_0\sqrt{1+\frac{B^2e^2}{4m^2\omega_0^2}}$$

$$=\pm\frac{Be}{2m}+\omega_0\left[1+\frac{B^2e^2}{8m^2\,\omega_0^2}\cdots\right]$$

$$\simeq+\frac{Be}{2m}+\omega_0$$

where we have neglected the second term in the paranthesis, as compared to unity.

$$\therefore \quad \Delta\omega=\omega-\omega_0=\pm\frac{Be}{2m}$$

But $\Delta\omega=2\pi\,\Delta\nu$

$$\therefore \quad \Delta\nu=\pm\frac{Be}{4\pi m}$$

33.31. University of Pittsburgh cyclotron has an oscillator frequency of 12×10^6 cycles/sec and a Dee radius of 21 in or 0.533 meter.

The magnetic induction required to accelerate deutrons is

$$B=\frac{2\pi\nu_0 m}{q}=\frac{(2\pi)(12\times10^6/\text{sec})(3.3\times10^{-27}\text{ kg})}{1.6\times10^{-19}\text{ coul}}$$

$$=1.6 \text{ webers/m}^2.$$

Deutron energy. $K_d=\frac{q^2B^2R^2}{2m}$

$$=\frac{(1.6\times10^{-19}\text{ coul})^2(1.6\text{ webers/meter}^2)^2\,(0.533\text{ meter})^2}{(2)(3.3\times10^{-27}\text{ kg})}$$

$$=2.8\times10^{-12}\text{ joule}.$$

$$=(2.8\times10^{-12}\text{ joule})\left(\frac{1\text{ eV}}{1.6\times10^{-19}\text{ joule}}\right)=17\text{ Mev}$$

(a) $$K=\frac{q^2B^2R^2}{2m}=2\pi^2\nu_0^2R^2m$$

$\therefore$ $$\frac{K_p}{K_d}=\frac{m_p}{m_d}=\frac{1}{2}$$

or $$K_p=\tfrac{1}{2}K_d=\tfrac{1}{2}\,(17\text{ Mev})=8.5\text{ Mev}$$

(b) $$B=\frac{2\pi\,\nu_0 m}{q}$$

$$\frac{B_p}{B_d}=\frac{m_p}{m_d}=\frac{1}{2}$$

$\therefore$ $$B_p=\tfrac{1}{2}B_d=\tfrac{1}{2}\,(1.6\text{ weber/meter}^2)=0.8\text{ weber/meter}^2$$

(c) $$K=\frac{q^2B^2R^2}{2m}=\frac{(1.6\times10^{-19}\text{ coul})^2(1.6\text{ weber/m}^2)^2\,(0.533\text{ meter})^2}{(2)(1.67\times10^{-27}\text{ kg})}$$

$$=(5.57\times10^{-12}\text{ joule})\left(\frac{1\text{ Mev}}{1.6\times10^{-13}\text{ joule}}\right)$$

$$=34.8\text{ Mev}$$

(d) $$\nu_0=\frac{Bq}{2\pi m}=\frac{(1.6\text{ weber/m}^2)\,(1.6\times10^{-19}\text{ coul})}{(2\pi)(1.67\times10^{-27}\text{ kg})}$$

$$=24.4\times10^6\text{ cycle/sec}=24.4\text{ Mc/sec}$$

(e) For (a) $$\frac{K_\alpha}{K_d}=\frac{m_\alpha}{m_d}=2$$

$\therefore$ $$K_\alpha=2K_d=(2)\,(17\text{ Mev})=34\text{ Mev}$$

For (b) $$\frac{B_\alpha}{B_d}=\frac{m_\alpha/q_\alpha}{m_d/q_d}=1$$

$\therefore$ $$B_\alpha=B_d=1.6\text{ weber/meter}^2.$$

For (c) $$K_\alpha=\frac{(2\times1.6\times10^{-19}\text{ coul})^2(1.6\text{ weber/m}^2)^2(0.533\text{ meter})^2}{(2)(6.68\times10^{-27}\text{ kg})}$$

$$=(5.57\times10^{-12}\text{ joule})(1\text{ Mev}/1.6\times10^{-13}\text{ joule})$$

$$=34.8\text{ Mev}.$$

For (d) $$\nu_0=\frac{Bq}{2\pi m}=\frac{(1.6\text{ weber/meter}^2)\,(2\times1.6\times10^{-19}\text{ coul})}{(2\pi)(6.68\times10^{-27}\text{ kg})}$$

$$=12.2\times10^6\text{ cycle/sec}$$

$$=12.2\text{ Mc/sec}.$$

33.32. Total path length traversed by a deutron is

$$L=\sum_{n=1}^{N} 2\pi R_n = 2\pi \int R_n dn \qquad \ldots(1)$$

where R_n is the radius of the n^{th} orbit and N=number of revolutions that the deutron makes during the acceleration process. If V is the potential between the Dees, then in each turn energy picked up by the deutron is 2 eV; the factor 2 arises due to the fact that the acceleration occurs twice for a given orbit. The energy gained after n revolutions is

$$K_n = 2eV_n = \frac{e^2B^2R^2_n}{2m}$$

$$\therefore \qquad R_n = 2\sqrt{mnV/eB^2} \qquad \ldots(2)$$

Using (2) in (1),

$$L=4\pi\sqrt{\frac{m\ V}{/eB^2}}\int_0^N \sqrt{n}dn = \frac{8\pi}{3}\left(\frac{mV}{eB^2}\right)^{\frac{1}{2}} N^{3/2}$$

Now, $$N=\frac{\text{total energy gained}}{\text{energy gained in each orbit}}=\frac{K}{2eV}$$

$$=\frac{17\times10^6 \text{ ev}}{(2)(8\times10^4 \text{ ev})}=106$$

where K=17Mev from textbook Example 7.

$$L=\frac{8\pi}{3}\sqrt{\frac{(3\ 3\times10^{-27}\text{ kg})(8\times10^4\text{ volts})}{(1.6\times10^{-19}\text{ coul})(1.6\text{ weber/meter}^2)^2}}$$
$$=(106)^{3/2}=232 \text{ meters.}$$

33.33. The oscillator frequency at the beginning of the acceleration cycle,

$$\nu_0=\frac{Bq}{2\pi m_0}$$

and at the end of the cycle

$$\nu=\frac{Bq}{2\pi m}$$

$$\therefore \qquad \frac{\nu_0}{\nu}=\frac{m}{m_0}=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{1}{\sqrt{1-\frac{(0.7C)^2}{C^2}}}=1.4$$

33.34. (*a*) Set $Bev = Ee$

$$\therefore \qquad B = \frac{E}{v} = \frac{E}{\sqrt{2K/m}}$$

$$= \frac{10^4 \text{ volts/meter}}{\sqrt{(2)(10^4 \text{ ev})(1.6 \times 10^{-19} \text{ joule/ev})/(9.1 \times 10^{-31} \text{ kg})}}$$

$$= 1.7 \times 10^{-4} \text{ weber/meter}^2.$$

Field of magnetic induction must be horizontal and to the left as one observes along v.

(*b*) yes.

For proton, $v = \frac{E}{B} = \frac{10^4 \text{ volts/meter}}{1.7 \times 10^{-4} \text{ weber/meter}^2}$

$$= 5.88 \times 10^7 \text{ meter/sec}$$

Kinetic energy of proton, $K = \frac{1}{2} Mv^2$

$$= \tfrac{1}{2}(1.67 \times 10^{-27} \text{ kg})(5.88 \times 10^7 \text{ meter/sec})^2$$

$$= (2.89 \times 10^{-12} \text{ joule})\left(\frac{1 \text{Mev}}{1.6 \times 10^{-13} \text{ joule}}\right) = 18 \text{ Mev}$$

Thus a proton of 18 Mev energy will pass through the given combination of electric and magnetic fields undeflected.

33.35. (*a*) $Bev = Ee$

The minimum speed is $v = \frac{E}{B} = \frac{1500 \text{ volts/meter}}{0.4 \text{ weber/meter}^2}$

$$= 3750 \text{ meter/sec.}$$

(*b*)

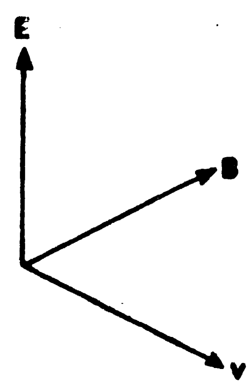


Fig. 33.35

SUPPLEMENTARY PROBLEMS

S.33.1. If a positive test charge q_0 is fired through a point and if a force **F** acts side ways on the moving charge and if the induction B is present at that point then the following relation is satisfied.

$$\mathbf{F}=q_0\mathbf{v}\times\mathbf{B}$$

Thus, the direction of **F** follows the rule for the vector product. The same result is obtained by the application of left-hand rule. On this basis particle 1 is positively charged as it is bending upward while B is acting into the page. Particle 2 is going straight undeviated. So it must be a neutral particle. Particle 3 is bending downward and must be negatively charged.

S.33.2. (*a*) Speed, $v=\dfrac{Ber}{m}$

$$=\frac{(0.41\times10^{-4}\ \text{weber/meter}^2)(1.6\times10^{-19}\ \text{coul})(6.4\times10^{6}\ \text{meters})}{(1.67\times10^{-27}\ \text{kg})}$$

$$=2.5\times10^{10}\ \text{meter/sec}$$

a value that exceeds the velocity of light. The fallacy is due to non-relativistic calculations.

(*b*)

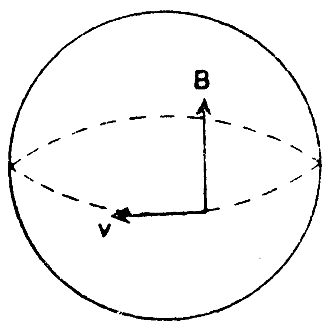


Fig. S.33.2

S.33.3. Kinetic energy of the electron, $K=1000$ ev

$$=(1000\ \text{eV})\left(1.6\times10^{-19}\ \frac{\text{joule}}{\text{ev}}\right)=1.6\times10^{-16}\ \text{joule}$$

$$K=\tfrac{1}{2}mv^2$$

$$\therefore\quad v=\sqrt{\frac{2K}{m}}=\sqrt{\frac{(2)(1.6\times10^{-16}\ \text{joule})}{(9.1\times10^{-31}\ \text{kg})}}$$

$$=1.88\times10^{7}\ \text{meter/sec.}$$

The electric field, $E=\frac{V}{d}=\frac{100 \text{ volt}}{0.02 \text{ meter}}=500$ volt/meter

If the electron moves undeviated then the force due to magnetic field and electric field must be equal.

$$Bev=Ee$$

$$\therefore \qquad B=\frac{E}{v}\frac{5000 \text{ volt/meter}}{1.88\times 10^7 \text{ meter/sec}}=2.66\times 10^{-4} \text{ weber/meter}^2$$

S.33.4. (*a*) The deflection is made zero if the electric force, $F_E=QE$, on the positive charge is equal and opposite to the magnetic force. If the velocity vector makes an angle θ with the induction vector $\boldsymbol{B}$ then the magnetic force exerted on charge Q is given by $F_B=QvB \sin\theta$, the direction of the force is always perpendicular to the plane determined by v and B.

Setting $F_B=F_E$

$QvB \sin\theta=QE$

$$\text{or} \qquad B=\frac{E}{v \sin\theta} \qquad \ldots(1)$$

The least value of B is given by the condition that $\sin\theta$ is maximum i.e. $\theta=90°$, so that

$$B=\frac{E}{v}$$

The direction of induction ought to be from east to west so that the magnetic force may act vertically down in opposition to the electric field.

(*b*) From Eq. (1) it is obvious that by varying the angle θ between the velocity vector and the induction vector, the magnitude of B would accordingly change. Thus, B is not unique for a given set of values of E and v if θ is not specified.

(*c*) Energy of proton,

$$K=3.1\times 10^5 \text{ ev}=(3.1\times 10^5 \text{ ev})(1.6\times 10^{-19} \text{ joule/ev})$$

$$=4.96\times 10^{-14} \text{ joule.}$$

$$\text{Velocity, } v=\sqrt{\frac{2K}{m}}=\sqrt{\frac{(2)(4.96\times 10^{-14} \text{ joule})}{(1.67\times 10^{-27} \text{ kg})}}$$

$$=7.7\times 10^6 \text{ meter/sec.}$$

$$B=\frac{E}{v}=\frac{1.9\times 10^5 \text{ volt/meter}}{7.7\times 10^6 \text{ meter/sec}}=2.47\times 10^{-2} \text{ weber/meter}^2.$$

(*d*) Equating the magnetic force to the centripetal force,

$$Bev=\frac{mv^2}{r}$$

or $$r=\frac{mv}{Be}=\frac{(1.67\times10^{-27}\text{ kg})(7.7\times10^{6}\text{ meter/sec})}{(2.47\times10^{-2}\text{ weber/meter})(1.6\times10^{-19}\text{ coul})}$$

$=3.25$ meter.

S.33 5. (*a*) Let an auxiliary resistance R_0 be connected in series with the galvanometer resistance R_g so that the equivalent resistance is, $R=R_0+R_g$.

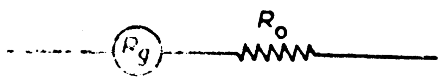


Fig. S.33.5 (a)

By $$V=iR$$

$$R=R_0+R_g=\frac{V}{i}$$

$$\therefore\quad R_0=\frac{V}{i}-R_g$$

$$R_0=\frac{1.0\text{ volt}}{1.62\times10^{-3}\text{ amp}}-75.3\text{ ohm}=542\text{ ohm}$$

(*b*) Let the auxiliary resistance R_0 be connected in parallel to R_g, then

$$i=i_g+i_0 \qquad ...(1)$$

$$i_gR_g=i_0R_0 \qquad ...(2)$$

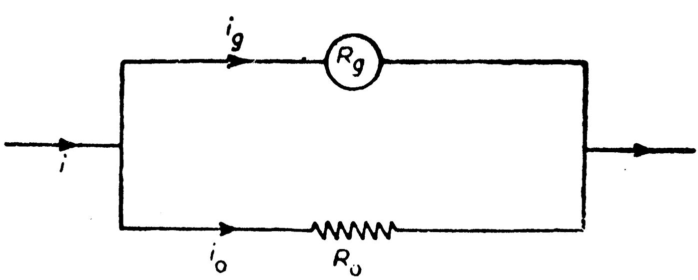


Fig. S.33.5 b

Eliminating i_0, we get,

$$R_0=\frac{R_gi_g}{i-i_g}=\frac{(75.3\text{ ohm})(1.62\times10^{-3}\text{ amp})}{(50\times10^{-3}\text{ amp}-1.62\times10^{-3}\text{ amp})}$$

$=2.52$ ohms.

S.33.6. (*a*) Hall electric field,

$$E_H=\frac{V}{d}=\frac{1.0\times10^{-5}\text{ volt}}{0.01\text{ meter}}=10^{-3}\text{ volt/meter}$$

$$E_H=-\mathbf{v}_d\times\mathbf{B}$$

or
$$v_d=\frac{E_H}{B}=\frac{10^{-3}\text{ volt/meter}}{1.5\text{ weber/meter}^2}$$
$$=6.7\times10^{-4}\text{ meter/sec}=0.067\text{ cm/sec.}$$

(*b*) Current density, $j=\dfrac{i}{A}$

$$=\frac{3\text{ amp}}{(10^{-2}\text{ meter})(10^{-5}\text{ meter})}=3\times10^{7}\text{ amp/meter}^2$$

The number of charge carriers per unit volume,

$$n=\frac{jB}{eE_H}=\frac{(3\times10^{7}\text{ amp/meter}^2)(1.5\text{ weber/meter}^2)}{(1.6\times10^{-19}\text{ coul})(10^{-3}\text{ volt/meter})}$$
$$=2.8\times10^{29}/\text{meter}^3$$
$$=2.8\times10^{23}/\text{cm}^3.$$

See textbook Fig. 33.10 (*b*).

For negative charge carriers, according to Fleming's left hand rule, face y will be at a lower potential than face x.

S.33.7. (*a*) $$\nu_0=\frac{qB}{2\pi m}=\frac{(1.6\times10^{-19}\text{ coul})(10^{-4}\text{ weber/meter}^2)}{(2\pi)(9.1\times10^{-31}\text{ kg})}$$
$$=2.8\times10^{6}\text{ cycles/sec}$$
$$=2.8\text{ Mc/sec.}$$

(*b*) Velocity of electron,

$$v=\sqrt{\frac{2K}{m}}=\sqrt{\frac{(2)(100\text{ ev}\times1.6\times10^{-19}\text{ joule/ev})}{(9.1\times10^{-31}\text{ kg})}}$$
$$=5.9\times10^{6}\text{ meter/sec}$$

Radius of curvature, $$r=\frac{mv}{Be}=\frac{(9.1\times10^{-31}\text{ kg})(5.9\times10^{6}\text{ meter/sec})}{(10^{-4}\text{ weber/meter}^2)(1.6\times10^{-19}\text{ coul})}$$
$$=0.33\text{ meter.}$$

34 AMPERE'S LAW

34.1. Radius of wire is

$$R=0.05 \text{ in}$$

$$=(0.05 \text{ in})\left(2.54\times10^{-2}\ \frac{\text{meter}}{\text{in.}}\right)=1.27\times10^{-3} \text{ meter}$$

Field B at the surface of the wire is

$$B=\frac{\mu_0\, i}{2\pi R}=\frac{(4\pi\times10^{-7} \text{ weber/amp-m})(50 \text{ amp})}{(2\pi)(1.27\times10^{-3} \text{ meter})}$$

$$=7.9\times10^{-3} \text{ weber/meter}^2.$$

34.2. Distance from the power line is

$$r=20 \text{ ft}=(20 \text{ ft})\left(0.3048\ \frac{\text{meter}}{\text{ft}}\right)=6 \text{ meters}$$

At the site of the compass,

$$B=\frac{\mu_0\, i}{2\pi r}=\frac{(4\pi\times10^{-7} \text{ weber/amp-m})(100 \text{ amp})}{(2\pi)(6 \text{ meters})}$$

$$=3.3\times10^{-6} \text{ weber/meter}^2$$

$$=0.033 \text{ gauss/cm}^2.$$

This value cannot be neglected compared to the horizontal component of earth's magnetic field of 0.2 gauss. Thus, there will be a serious interference with the compass reading.

34.3. Near the electron,

$$B=\frac{\mu_0\, i}{2\pi r}=\frac{(4\pi\times10^{7} \text{ weber/amp-m})(50 \text{ amp})}{(2\pi)(0.05 \text{ meter})}$$

$$=2\times10^{-4} \text{ weber/meter}^2.$$

(*a*) Force on the electron is $F=q\mathbf{v}\times\mathbf{B}=qvB\sin\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{B}$. Set $\theta=90°$.

$$F=qvB=(1.6\times10^{-19} \text{ coul})(10^{7} \text{ meter/sec})(2\times10^{-4} \text{ weber/meter}^2)$$

$$=3.2\times10^{-16} \text{ nt, parallel to current.}$$

(*b*) Assuming that the velocity is parallel to the current, a force equal to 3.2×10^{-16} nt would act radially outward.

(*c*) $\theta=0$. Hence, $F=0$.

34.4. (*a*) Consider a rectangular loop *abcd* of length l enclosing a portion of the conductor, (Fig. 34.4 (*a*).

Applying Ampere's law,

$$\oint \mathbf{B} . d\mathbf{l} = \mu_0 i_0$$

Now, the contribution to the integral from the path *ad* and *bc* is zero as it is perpendicular to **B**. Furthermore,

$$i_0 = inl$$

Also, **B** and $d\mathbf{l}$ are parallel along the paths *ba* and *dc* and as their contributions are equal,

$$2Bl = \mu_0 \; in \; l$$

or $$B = \frac{\mu_0 ni}{2}.$$

(*b*) At every field point the horizontal component of B alone will be reinforced and the vertical component will get cancelled from considerations of symmetry and the direction of B will be as indicated in Fig. 34.4 (*b*).

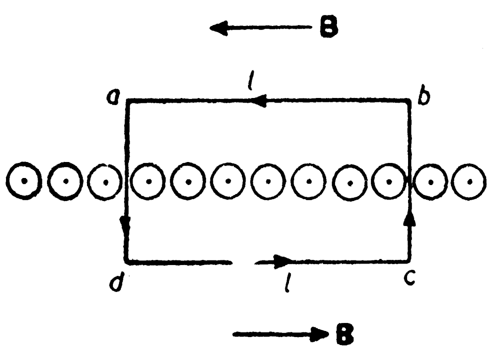


Fig. 34.4, (a)

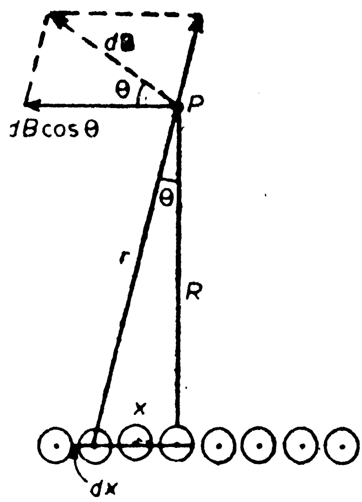


Fig. 34.4. (b)

Consider an infinitesimal width dx through which a current di flows. Then

$$di = ni \; dx. \qquad ...(1)$$

The field contributed by di at P is

$$dB = \frac{\mu_0 di}{2\pi r} \qquad ...(2)$$

which is the differential form for the magnetic induction for a long straight wire.

$$r = R \sec\theta \qquad \text{...(3)}$$

$$x = R \tan\theta$$

$$dx = R \sec^2\theta \, d\theta \qquad \text{...(4)}$$

Using (1), (3) and (4) in (2)

$$dB = \frac{\mu_0 ni \sec\theta \, d\theta}{2\pi}. \qquad \text{...(5)}$$

The induction B at point P is given by the integral,

$$B = \int dB \cos\theta = \frac{\mu_0 \, ni}{2\pi} \int_{-\pi/2}^{\pi/2} d\theta = \frac{\mu_0 ni}{2}$$

where use has been made of (5).

34.5. (a) By Ampere's law,

$$\oint \mathbf{B}.d\mathbf{l} = \mu_0 \, i_0 \qquad \text{...(1)}$$

where i_0 is the current flowing through the body of the conductor.

Now,

$$i_0 = \frac{i\pi \, (r^2 - a^2)}{\pi \, (b^2 - a^2)} = \frac{i \, (r^2 - a^2)}{(b^2 - a^2)} \qquad \text{...(2)}$$

Evaluating the integral in (1) and using (2) in the right hand side,

$$(B)(2\pi r) = \mu_0 \, i \frac{(r^2 - a^2)}{(b^2 - a^2)}$$

or
$$B = \frac{\mu_0 \, i \, (r^2 - a^2)}{2\pi(b^2 - a^2) \, r} \qquad \text{...(3)}$$

This gives, $B = \dfrac{\mu_0 \, ir}{2\pi b^2}$, for $a = 0$.

This is the expected result for B at a distance r from the center of a long cylindrical wire of radius b, where $r < b$.

(b) The general behaviour of B (r) from $r = 0$ to $r = \infty$ is shown in Fig 34.5. For $r < a$, as there is no current flowing, B will be zero. For $a < r < b$, B is given by (3) and for $r > b$, B is given by

$$B = \frac{\mu_0 \, i}{2\pi r}$$

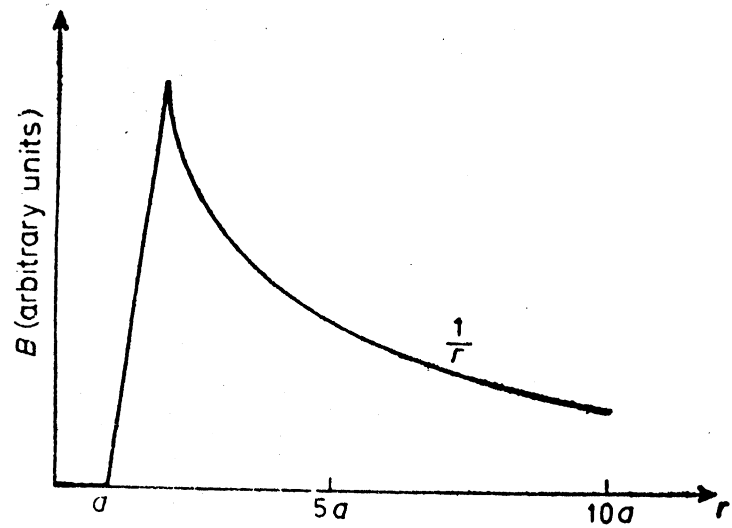


Fig. 34.5

34.6. (*a*) The net current passing through the conductor bounded by the closed path corresponds to that flowing through the liner cylinder above.

Hence, by Ampere's theorem

$$\oint \mathbf{B}d\mathbf{l}=(B)(2\pi r)=\mu_0 i\,\frac{(\pi r^2)}{\pi a^2}$$

or $$B=\frac{\mu_0 i r}{2\pi a^2}$$

(*b*) Here current through outer cylinder does not contribute to B.

$$\oint \mathbf{B}.d\mathbf{l}=(B)(2\pi r)=\mu_0 i$$

or $$B=\frac{\mu_0 i}{2\pi r}$$

(*c*) Here current through both the cylinders contribute to B.

$$\oint \mathbf{B}.d\mathbf{l}=(B)(2\pi r)=\mu_0 i-\mu_0 i\,\frac{\pi(r^2-b^2)}{\pi^2(c^2-b^2)}$$

or $$B=\cancel{\frac{\mu_0 i\,(c^2-r^2)}{2\pi r\,(c^2-b^2)}}\qquad \frac{\mu_0 i}{2\pi r}\left(\frac{c^2-r^2}{c^2-b^2}\right)$$

(*d*) As the net current flowing through the closed path is zero, $B=0$.

34.7. By Ampere's theorem applied to the interior of the wire at a radial distance r,

$$\oint \mathbf{B}.d\mathbf{l}=\mu_0 i'$$

Where $i'=i(r/a)^2$=current inside radius r. Current i is the current in the entire wire of radius a.

$$B(r)=\frac{\mu_0 i}{2\pi r}\left(\frac{r}{a}\right)^2=\frac{\mu_0 ir}{2\pi a^2}$$

Since the surface S is normal to induction, the flux is given by,

$$\phi=\int B(r)\,dA=\int B(r)\,l\,dr$$

where l is the length of the wire.

$$\phi=l\int_0^a \frac{\mu_0 ir}{2\pi a^2}\,dr=\frac{\mu_0 il}{4\pi}$$

Hence, the magnetic flux per meter of wire is

$$\frac{\phi}{l}=\frac{\mu_0 i}{4\pi}=\frac{(4\pi\times10^{-7}\text{ weber/amp-meter})(10\text{ amp})}{4\pi}$$

$$=10^{-6}\text{ weber/meter.}$$

34.8. (*a*) By Problem 34.7, in the interior of each wire ($r<a$), the flux per meter is

$$\frac{\phi}{l}=\frac{\mu_0 i}{4\pi}=\frac{(4\pi\times10^{-7}\text{ weber/amp-m})(10\text{ amp})}{4\pi}$$

$$=10^{-6}\text{ weber/meter.}$$

For antiparallel currents, the inductions are additive

Hence, the flux/meter in the region $r<a$, for the two wires is 2×10^{-6} weber/meter, where a=0.127 cm is the radius of the wire. For the space between the two wires the flux is calculated from,

$$\phi=\frac{2\mu_0 i}{2\pi}\int_a^d\frac{dx}{x}=\frac{\mu_0 i}{\pi}\ln x\Big|_a^d$$

$$=\frac{\mu_0 i}{\pi}\ln\left(\frac{d}{a}\right)$$

$$=\frac{(4\pi\times10^{-7}\text{ weber/amp-meter})(10\text{ amp})}{\pi}\ln\left(\frac{2.0}{0.127}\right)$$

$$=11\times10^{-6}\text{ weber/meter}$$

where d is the distance between the centers of the wires.

Therefore, the flux per meter that exists in the space between the axes of the wires is

$$\frac{\phi_{(total)}}{l}=(2\times10^{-6}+11\times10^{-6})=13\times10^{-6}\text{ weber/meter}\cdot$$

(*b*) Fraction of flux that lies inside the wires is

$$f=\frac{2\times10^{-6}\text{ weber/meter}}{13\times10^{-6}\text{ weber/meter}}=0.16$$

(*c*) Since the inductions get cancelled for parallel currents, the flux/meter is equal to zero.

34.9. Field of a long wise is $B=\frac{\mu_0 i}{2\pi r}$.

$$\therefore \qquad r=\frac{\mu_0 i}{2\pi B}=\frac{(4\pi\times10^{-7}\text{ weber/amp-meter})(100\text{ amp})}{(2\pi)(50\times10^{-4}\text{ weber/meter}^2)}$$

$$=4\times10^{-3}\text{ meter}=4.0\text{ mm.}$$

B will be zero along a line parallel to the wire and 4.0 mm distant. Suppose the current is horizontal and in the direction of the observer, and the external field pointing horizontally from left to right, then the line would be directly above the wire.

34.10. The current at *a* out of the plane of paper produces magnetic induction $\mathbf{B}_1$, at *P* at distance *r*, in a direction perpendicular to r_1 and is indicated by the arrow in Fig. 34.10 (*a*). The current at *b* into the plane or paper produces magnetic induction $\mathbf{B}_2$ at *P* at distance r_2 in a direction perpendicular to r_2. Resolve $\mathbf{B}_1$ and $\mathbf{B}_2$ in a direction parallel to *R* and perpendicular to it. Because $\mathbf{B}_1$ and $\mathbf{B}_2$ are equal in magnitude and are symmetrically oriented about *R*, the perpendicular components get cancelled and the parallel components are reinforced.

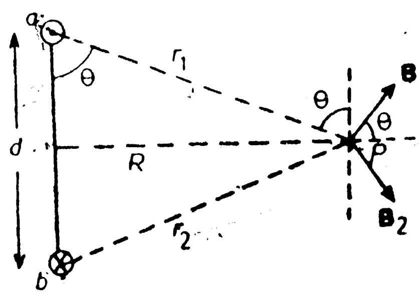


Fig. 34.10 (a)

$$B=B_1\cos\theta+B_2\cos\theta=2B_1\cos\theta$$

$$=\frac{2\mu_0 i}{2\pi r_1}\,\frac{d/2}{r_1}=\frac{\mu_0 i d}{2\pi r_1^2}$$

where we have used the fact that $\cos\theta=\frac{d}{2r_1}$

Further, $r_1^2=\frac{d^2}{4}+R^2=\frac{d^2+4R^2}{4}$

$$\therefore \qquad B=\frac{2\mu_0 i d}{\pi(d^2+4R^2)}$$

34.11. Choose the x and y axes parallel to the sides of the square. The magnetic induction B_1, B_2, B_3 and B_4 due to currents 1, 2, 3, and 4 respectively, at P, the center of the square, are indicated in Fig. 34-25 (a).

As the currents are equal and the point P is located at equal distance from the site of the currents,

$$B_1 = B_2 = B_3 = B_4$$

Resolve B_1, B_2, B_3 and B_4 along x and y axes; we note that the resultant of x-components vanishes. The y-components reinforce and the net induction is

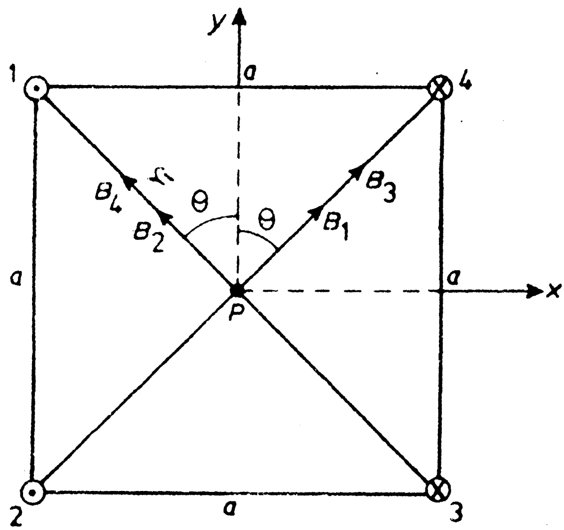


Fig. 34.11

$$B = 4B_1 \cos\theta \qquad \ldots(1)$$

$$B_1 = \frac{\mu_0 i}{2\pi r_1} \qquad \ldots(2)$$

where r_1 is half the diagonal.

$$r_1 = \frac{a}{\sqrt{2}} \qquad \ldots(3)$$

$$\cos\theta = \cos 45° = 1/\sqrt{2} \qquad \ldots(4)$$

$$\therefore \qquad B = \frac{4\mu_0 i \cos\theta}{2\pi a/\sqrt{2}}$$

$$= \frac{(4)(4\pi \times 10^{-7}\ \text{weber-amp/meter}(20\ \text{amp})(1/\sqrt{2})}{(2\pi)(0.2\ \text{meter})/\sqrt{2}}$$

$= 8 \times 10^{-5}$ weber/meter^2, along y-axis.

34.12. We wish to calculate the force per meter acting on wire 2 due to wires 1, 3 and 4. Since the currents in wires 1 and 2 are parallel, the force F_{21} per meter on wire 2 due to 1 is attractive and is directed up, the magnitude being given by

$$F_{21}=\frac{\mu_0 i_1 i_2}{2\pi a}$$

$$=\frac{(4\pi\times10^{-7}\text{ weber-amp/m})(20\text{ amp})^2}{(2\pi)(0.2\text{ meter})}$$

$$=4\times10^{-4}\text{ nt.}$$

The current in wire 2 and 3 being anti-parallel, the force of 3 on 2 will be repulsive and acts towards left.

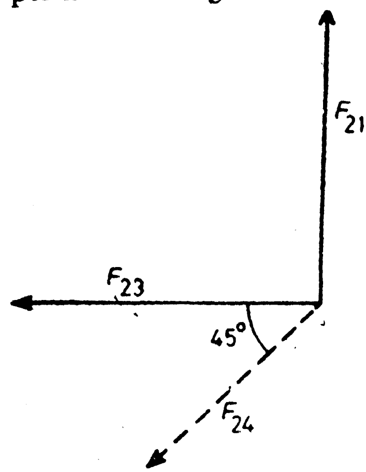


Fig. 34.12

$$F_{23}=\frac{\mu_0 i_2 i_3}{2\pi a}=\frac{(4\pi\times10^{-7}\text{ weber-amp/m})(20\text{ amp})^2}{(2\pi)(0.2\text{ meter})}$$

$$=4\times10^{-4}\text{ nt.}$$

Similarly, the force between wires 4 and 2 will be repulsive, along the diagonal joining 4 and 2.

$$F_{24}=\frac{\mu_0 i_2 i_4}{2\pi(\sqrt{2}a)}=\frac{(4\pi\times10^{-7}\text{ weber-amp/m})(20\text{ amp})^2}{(2\pi)(\sqrt{2})(0.2\text{ meter})}$$

$$=2\sqrt{2}\times10^{-4}\text{ nt.}$$

The x-component of the net force

$$F_x=F_{23}+F_{24}\cos 45°=4\times10^{-4}\text{ nt}+2\times10^{-4}\text{ nt}=6\times10^{-4}\text{ nt}$$

The y-component of the net force

$$F_y=F_{21}-F_{24}\sin 45°=4\times10^{-4}\text{ nt}-2\times10^{-4}\text{ nt}=2\times10^{-4}\text{ nt}$$

The magnitude of the force is $F=\sqrt{F_x^2+F_y^2}$

$$=\sqrt{(6\times10^{-4}\text{ nt})^2+(2\times10^{-4}\text{ nt})^2}=2\sqrt{10}\times10^{-4}\text{ nt.}$$

It is directed at an angle $\theta=\tan^{-1}\dfrac{F_x}{F_y}=\tan^{-1}\dfrac{6\times10^{-4}\text{ nt}}{2\times10^{-4}\text{ nt}}$

$$=\tan^{-1}3=71.6°\text{ with the } y\text{-axis towards left.}$$

34.13. The longer sides of the loop alone contribute to the force. In the left vertical branch of the loop the current is parallel to that in the long wire. Hence, the force is attractive while in the right branch the current is anti-parallel and hence repulsive. The resultant force acting on the loop is

$$F=\frac{\mu_0 i_1 i_2 l}{2\pi a}-\frac{\mu_0 i_1 i_2 l}{2\pi(a+b)}=\frac{\mu_0 i_1 i_2 l b}{2\pi a(a+b)}$$

$$=\frac{(4\pi\times10^{-7}\text{ weber-amp/m})(30\text{ amp})(20\text{ amp})(0.3\text{ meter})(0.08\text{ meters})}{(2\pi)(0.01\text{ meter})(0{,}01\text{ meter}+0.08\text{ meter})}$$

$=3.2\times10^{-3}$ nt, directed towards the long wire.

34.14. Field on the axis of a circular loop is

$$B=\frac{\mu_0\, iR^2}{2\,(R^2+x^2)^{3/2}}$$

Let there be n turns per unit length of the solenoid. Consider an elementary length of the solenoid. The number of turns in the length dx is ndx. The flux denstiy at a given point on the axis set up by a current i in the element of length dx is

$$dB=\frac{\mu_0\, niR^2\, dx}{2\,(R^2+x^2)^{3/2}}$$

With the change of independent variable θ defined by

$$x=R\cot\theta$$

$$dx=-R\,\text{cosec}^2\,\theta\, d\theta$$

$$B=\frac{\mu_0\, niR^2}{2}\int_{-\infty}^{\infty}\frac{dx}{(R^2+x^2)^{3/2}}$$

$$=\frac{\mu_0\, niR^2}{2}\int_{\pi}^{0}\frac{(-R\,\text{cosec}^2\,\theta\, d\theta)}{(R^2+R^2\cot^2\theta)^{3/2}}$$

$$=-\frac{\mu_0\, ni}{2}\int_{\pi}^{0}\sin\theta\, d\theta=\frac{\mu_0\, ni}{2}\cos\theta\Big|_{\pi}^{0}=\mu_0\, ni$$

34.15. Induction set up at P due to current i flowing through one of the wires at the sides of the square in the elementary length dx (Fig 34.15) is,

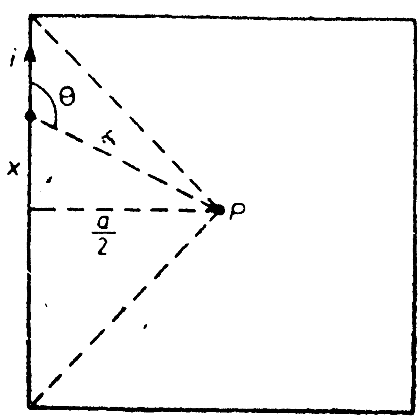


Fig. 34.15

$$dB=\frac{\mu_0\, idx\sin\theta}{4\pi r^2}$$

$$r^2=x^2+\frac{a^2}{4}$$

$$\sin\theta=\frac{a/2}{\sqrt{x^2+a^2/4}}$$

$$B=\frac{\mu_0\, i}{4\pi}\int\frac{a}{2}\;\frac{dx}{(x^2+a^2/4)^{3/2}}$$

Set $x=a/2 \cot\theta$

$$dx=-a/2 \operatorname{cosec}^2\theta\, d\theta$$

We get, $$B=-\frac{\mu_0 i}{2\pi a}\int_{135°}^{45°}\sin\theta\, d\theta=\frac{\mu_0 i}{2\pi a}\cos\theta\Big|_{135°}^{45°}$$

$$=\frac{\mu_0 i}{\sqrt{2}\,\pi a}$$

Since B due to the four wires at the sides of the square is set up in the same direction, the value of B at the center is given by,

$$B_{(Resultant)}=4B=\frac{4\mu_0 i}{\sqrt{2}\,\pi a}=\frac{2\sqrt{2}\,\mu_0 i}{\pi a}.$$

34.16. (*a*) We first consider the induction at P due to current i flowing through the top side of the square, Fig. 34.16. As the point P is symmetrically situated about the square, the induction due to a right conductor of length (top side) is

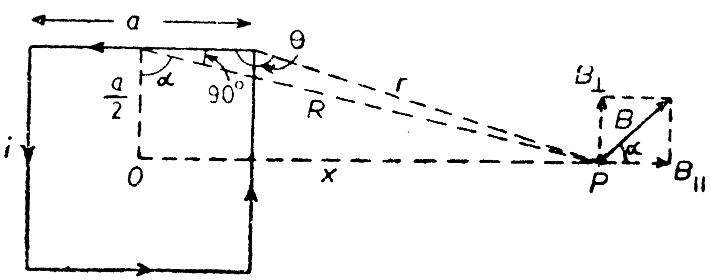


Fig. 34.16

$$B=\frac{2\mu_0 i\cos\theta}{4\pi R}=\frac{\mu_0 ia}{4\pi Rr} \qquad \ldots(1)$$

where we have used the relation

$$\cos\theta=\frac{a}{2r} \qquad \ldots(2)$$

Resolve B into two mutually perpendicular components, $B_{\|}$ along the axis of the loop and $B_{\perp}$ perpendicular to it. It is seen that $B_{\perp}$ gets cancelled by the contribution of the current flowing through opposite side of the square. On the other hand, the components $B_{\|}$ are reinforced.

$$B_{\|}=B\cos\alpha \qquad \ldots(3)$$

Thus, the induction at the point P due to the entire loop is given by

$$B_{(loop)}=4B_{\parallel}=4B\cos\alpha=\frac{\mu_0\,ia}{\pi rR}\cos\alpha \qquad \ldots(4)$$

where use has been made of (1) and (3).

$$\cos\alpha=\frac{a}{2R} \qquad \ldots(5)$$

$$R^2=x^2+\frac{a^2}{4} \qquad \ldots(6)$$

$$r=\left(R^2+\frac{a^2}{4}\right)^{\frac{1}{2}}=\left(x^2+\frac{a^2}{2}\right)^{\frac{1}{2}} \qquad \ldots(7)$$

Using (5), (6) and (7) in (4) and simplifying,

$$B_{(loop)}=\frac{4\mu_0\,ia^2}{\pi(4x^2+a^2)(4x^2+2a^2)^{1/2}} \qquad \ldots(8)$$

(*b*) Set $x=0$.

$$B_{(loop)}=\frac{2\sqrt{2}\,\mu_0 i}{\pi a}$$

which is identical with the result of Problem 34.15.

(*c*) For $x \gg a$, the terms a^2 and $2a^2$ in the denominator of formula (8) can be neglected. Then (8) reduces to

$$B=\frac{\mu_0\,ia^2}{2\pi x^3} \qquad \ldots(9)$$

On comparing (9) with formula (10) for the magnetic field at distant points along axis

$$B=\frac{\mu_0\mu}{2\pi x^3} \qquad \ldots(10)$$

We conclude that the square loop behaves like a dipole moment, with the dipole moment given by

$$\mu=ia^2.$$

34.17. (*a*) Consider a typical current element dy. The magnitude of the contribution dB of this element to the magnetic field at P is found from Biot-Savart law and is given by

$$dB=\frac{\mu_0\,idy\sin\theta}{4\pi\,r^2} \qquad \ldots(1)$$

Since the directions of the contributions dB at point P for all elements are identical viz., at right angles into the plane of figure, the resultant field is obtained by simply integrating (1).

$$B=\int dB=\frac{\mu_0 i}{4\pi}\int_{-l/2}^{l/2}\frac{\sin\theta\, dy}{r^2}$$

But sin $\theta=\frac{R}{r}$

$$\therefore \quad B=\frac{\mu_0 iR}{4\pi}\int_{-l/2}^{l/2}\frac{dy}{r^3}=\frac{\mu_0 iR}{4\pi}\int_{-l/2}^{l/2}\frac{dy}{(y^2+R^2)^{3/2}}$$

$$=\frac{\mu_0 iR}{4\pi}\frac{1}{R^2}\frac{y}{\sqrt{y^2+R^2}}\Bigg|_{-l/2}^{l/2}$$

$$=\frac{\mu_0 il}{2\pi R\,(l^2+4R^2)^{1/2}}$$

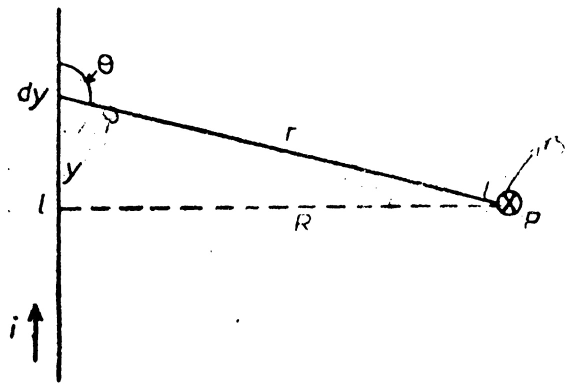


Fig 34.17

(*b*) If $l\to\infty$ then the term $4R^2$ can be neglected and $B\to\frac{\mu_0 i}{2\pi R}$ a result which is identical with that expected for a current in a long straight wire.

34.18. The magnetic induction at C due to a current element dx is given by

$$dB=\frac{\mu_0 i\sin\theta dx}{4\pi r^2}$$

(*a*) Here $\theta=0$ for the left straight branch and $\theta=180°$ for the right straight branch. In either case $B=0$.

(*b*) Here $\theta=90°$ since the radius of the semi-circle is perpendicular to current element. Setting $r=R$

$$B=\frac{\mu_0 \, i \sin 90°}{4\pi R^2}\int_0^{\pi R} dx=\frac{\mu_0 \, i}{4R}$$

(*c*) Since the straight portions of the wire do not contribute to B, the value of B due to the entire wire is the same as that for the semi-circle viz., $\frac{\mu_0 \, i}{4R}$

34.19. (*a*) The magnetic induction at the center of the circle due to current element dx in one side of the polygon is

$$dB'=\frac{\mu_0 i \sin \theta \, dx}{4\pi r^2}$$

Since dB' points in the same direction viz., into the plane of figure at right angles for all the current elements the contributions to the field is obtained by integrating the above expression and multiplying the result by the number of sides (Fig 34.19).

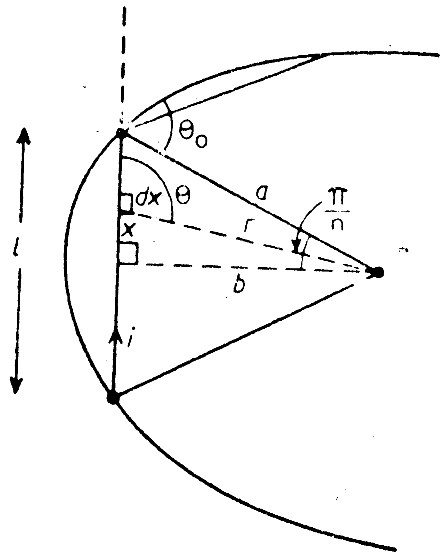


Fig. 34.19

$$\sin \theta=\frac{b}{r}$$

$$dB'=\frac{\mu_0 i b dx}{4\pi r^3}=\frac{\mu_0 i b dx}{4\pi(x^2+b^2)^{3/2}}$$

$$\therefore \quad B'=\frac{\mu_0 i b}{4\pi}\int_{-l/2}^{l/2}\frac{dx}{(x^2+b^2)^{3/2}}=\frac{2\mu_0 i b}{4\pi}\int_0^{l/2}\frac{dx}{(x^2+b^2)^{3/2}}$$

But $$\int_0^{l/2} \frac{dx}{(x^2+b^2)^{3/2}} = \frac{x}{b^2\sqrt{x^2+b^2}}\Bigg|_0^{l/2} = \frac{l/2}{b^2\sqrt{(l^2/4+b^2)}}$$

$$=\frac{1}{b^2}\sin\frac{\pi}{n}$$

$$\therefore \qquad B'=\frac{\mu_0 ib}{2\pi}\,\frac{1}{b^2}\sin\frac{\pi}{n}=\frac{\mu_0 i}{2\pi b}\sin\frac{\pi}{n}$$

$$b=a\cos\frac{\pi}{n}$$

$$\therefore \qquad B'=\frac{\mu_0 i}{2\pi a}\tan\frac{\pi}{n}$$

The magnetic induction at the center of the circle due to n sides is

$$B=nB'=\frac{\mu_0 in}{2\pi a}\tan\frac{\pi}{n}$$

(*b*) As $n\to\infty$, $\tan\frac{\pi}{n}\to\frac{\pi}{n}$

$$B\to\frac{\mu_0 i}{2a}$$

a result which is identical for B of a circular loop. This is reasonable since as $n\to\infty$, the polygon→circle.

34 20. (*a*) Using the result of Problem 34.17 we find at the center of the rectangle, the induction

$$B_1=\frac{2\mu_0 il}{2\pi(d/2)\sqrt{l^2+d^2}}=\frac{2\mu_0 il}{\pi d\sqrt{l^2+d^2}}$$

where the factor 2 in the numerator takes care of the two longer sides of the rectangle each of length l. Similarly, for the shorter sides each of width d,

$$B_2=\frac{2\mu_0 id}{\pi l\sqrt{l^2+d^2}}$$

$$\therefore \qquad B=B_1+B_2=\frac{2\mu_0 i}{\pi\sqrt{l^2+d^2}}\left(\frac{l}{d}+\frac{d}{l}\right)=\frac{2\mu_0 i(l^2+d^2)^{1/2}}{\pi ld}$$

(*b*) If $l \gg d$

$$B=\frac{2\mu_0 i\sqrt{l^2+d^2}}{\pi ld}=\frac{2\mu_0 il}{\pi ld}=\frac{2\mu_0 i}{\pi d}$$

where we have neglected d^2 under the radical. This is the expected result for the value of B mid-way between two parallel long condutors separated by distance d.

34.21. The field at any point P on the axis due to the left coil is

$$B_1 = \frac{\mu_0 i N R^2}{2(R^2+x^2)^{3/2}}$$

where x is measured from the center of the coil.

As the distance is being measured from P, the middle point of the separation of coils, replace x by $x+(R/2)$ in the above expression. Similarly, for the second coil, replace x by $(R/2)-x$. Hence, the resultant,

$$B = \frac{\mu_0 i N R^2}{2}\left\{\frac{1}{[R^2+(R/2+x)^2]^{3/2}} + \frac{1}{[R^2+(R/2-x)^2]^{3/2}}\right\} \quad \ldots(1)$$

Fig. 34.21 shows the plot of B versus x for the given data.

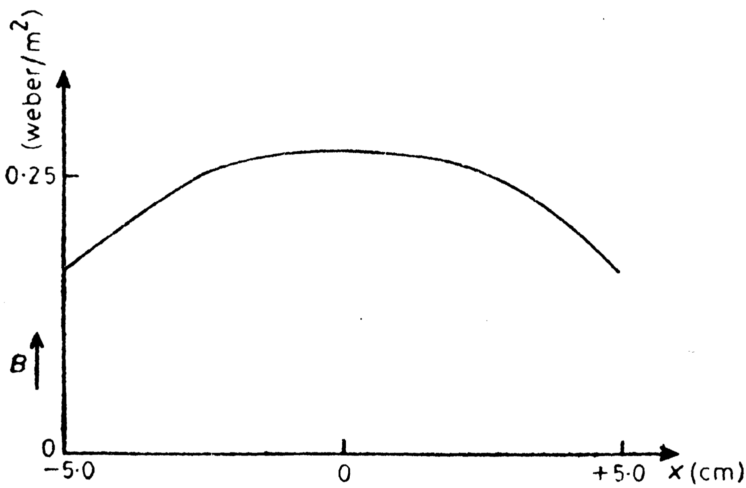


Fig. 34.21

34.22. The resultant field due to the two coils at any point a distance x along the axis from the center of the left coil is

$$B = \frac{\mu_0 i N R^2}{2}\left\{\frac{1}{(R^2+x^2)^{3/2}} + \frac{1}{[R^2+(z-x)^2]^{3/2}}\right\} \quad \ldots(1)$$

For the purpose of investigating the variation of B with x we can ignore the constant factors $\frac{1}{2}\mu_0 i N R^2$. We then find upon differentiating B with respect to x,

$$\frac{\partial B}{\partial x} = -\frac{3}{2}\,\frac{2x}{(R^2 \times x^2)^{5/2}} - \frac{3}{2}\,\frac{2(z-x)(-1)}{[R^2+(z-x)^2]^{5/2}} \quad \ldots(2)$$

Set $x=R/2$ and $z=R$. Then we find upon substituting these values in the above expression that $\partial B/\partial x=0$. Differentiating (2) once again with respect to x,

$$\frac{\partial^2 B}{\partial x^2}=-3\frac{\partial}{\partial x}\frac{x}{(R^2+x^2)^{5/2}}+3\frac{\partial}{\partial x}\frac{(z-x)}{[R^2+(z-x)^2]^{5/2}}$$

$$=-3\left[\frac{(R^2+x^2)^{5/2}-x\,(5/2)(2x)(R^2+x^2)^{3/2}}{(R^2+x^2)^5}\right]$$

$$+3\left[\frac{R^2+(z-x)^2(5/2)(-1)-(z-x)(5/2(2)(z-x)(-1)[R^2+(z-x)^2]^{3/2}}{[(R^2+(z-x)^2]^5}\right].$$

Set $x=R/2$ and $z=R$, then

$$\frac{\partial^2 B}{\partial x^2}=0.$$

34.23. The direction of B at both the points a and b is perpendicular to the plane of paper upwards (Fig 34.23).

The field at a is contributed by there paths, I, II and III and is calculated as follows:

$$B_a=B_I+B_{II}+B_{III}$$

Following the methods of Problem 34.17,

$$B_I=\frac{\mu_0\, i}{4\pi}\int_0^\infty\frac{R\,dx}{(x^2+R^2)^{3/2}}=\frac{\mu_0\, i}{4\pi R}\cos\theta\Big|_{90^\circ}^{0}=\frac{\mu_0\, i}{4\pi R}$$

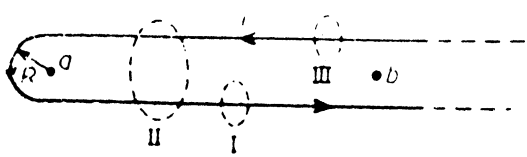


Fig. 34.23

Following the method of Problem 34.18,

$$B_{II}=\frac{\mu_0\, i}{4R}$$

Also, $B_{III}=\dfrac{\mu_0\, i}{4\pi R}$

$$\therefore\qquad B_a=\frac{\mu_0\, i}{4R}\left(\frac{1}{\pi}+1+\frac{1}{\pi}\right)=\frac{\mu_0\, i\,(2\pi+1)}{4\pi R}$$

$$=\frac{(4\pi\times10^{-7}\text{ weber/amp-m})(10\text{ amp})(2\pi+1)}{(4\pi)(0.005\text{ meter})}$$

$$=1.03\times10^{-3}\text{ weber/meter}^2.$$

The field at b is mainly contributed by the paths I and III, the contribution by the path II being zero.

$$B_b=B_I+B_{III}$$

Here, $B_I=\dfrac{\mu_0 i}{4\pi}\displaystyle\int_{-\infty}^{\infty}\frac{R\,dx}{(x^2+R^2)^{3/2}}=\frac{\mu_0 i}{2\pi R}$

Similarly, $B_{III}=\dfrac{\mu_0 i}{2\pi R}$

$$\therefore\quad B_b=\frac{\mu_0 i}{2\pi R}+\frac{\mu_0 i}{2\pi R}=\frac{\mu_0 i}{\pi R}=\frac{(4\pi\times10^{-7}\text{ weber/amp-m})(10\text{ amp})}{(\pi)(0.005\text{ meter})}$$

$$=8\times10^{-4}\text{ weber/meter}^2.$$

34.24. The field due to a straight conductor carrying current i at the point P at distance R is given by,

$$B=\frac{\mu_0 iR}{4\pi}\int_{x_1}^{x_2}\frac{dx}{(x^2+R^2)^{3/2}}$$

$$=\frac{\mu_0 i}{4\pi R}\left.\frac{x}{\sqrt{x^2+R^2}}\right|_{x_1}^{x_2}\qquad\text{...(1)}$$

It is seen that contribution to B at P from the left path (B_1) and top path (B_2) will be identical. Similarly the contribution to B at P from the right path (B_3) and the bottom path (B_4) will be identical, the field in each case pointing in the same direction viz., into the paper perpendicular to the figure.

$$B=B_1+B_2+B_3+B_4\qquad\text{...(2)}$$

$$B_1=B_2=\frac{\mu_0 i}{4\pi(a/4)}\left.\frac{x}{\sqrt{x^2+a^2/16}}\right|_{-a/4}^{3a/4}\qquad\text{...(3)}$$

where we have set, $R=\dfrac{a}{4}$.

$$\therefore\quad B_1=B_2=\frac{\mu_0 i}{\pi a}\left(\frac{3}{\sqrt{10}}+\frac{1}{\sqrt{2}}\right)\qquad\text{...(4)}$$

Also, $$B_3=B_4=\frac{\mu_0\, i}{4\pi(3a/4)}\left.\frac{x}{\sqrt{x^2+\frac{9a^2}{16}}}\right|_{-a/4}^{3a/4} \quad \text{.. (5)}$$

where we have set $R=3a/4$.

$$\therefore \quad B_3=B_4=\frac{\mu_0 i}{3\pi a}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{10}}\right) \quad \text{...(6)}$$

Using (4) and (6) in (2),

$$B=\frac{2\ \mu_0\, i}{3\pi a}(\sqrt{10}+2\sqrt{2})$$

$$=\frac{(2)(4\pi\times10^{-7}\text{ weber/amp-m})(10\text{ amp})(\sqrt{10}+\sqrt{22})}{(3\pi)(0.08\text{ meter})}$$

$$=2\times10^{-4}\text{ weber/meter}^2.$$

?4.25. (*a*) Consider a ring of radius r and width dr concentric with the disk. Then the charge associated with the ring is

$$dq=\frac{(2\pi r dr)}{\pi R^2}\ q=\frac{2qrdr}{R^2} \quad \text{...(1)}$$

The current is the rate at which charge passes any point on the ring and is given by

$$i=\nu dq \quad \text{...(2)}$$

where, $$\nu=\frac{\omega}{2\pi} \quad \text{...(3)}$$

is the rotational frequency of the disk.

$$dB=\frac{\mu_0 i}{2r}=\frac{\mu_0\, \nu dq}{2r} \quad \text{...(4)}$$

where use has been made of (2). Using (1) and (3) in (4).

$$dB=\frac{\mu_0\, \omega q dr}{2\pi R^2}$$

$$B=\frac{\mu_0\, \omega\, q}{2\pi R^2}\int_0^R dr=\frac{\mu_0\, \omega q}{2\pi R}.$$

(*b*) $$i=\nu dq=\frac{\omega}{2\pi}\ \frac{2qr\, dr}{R^2}=\frac{\omega q r dr}{\pi R^2}.$$

Contribution to the magnetic moment is

$$d\mu=iA=\frac{(\omega q r dr)}{\pi R^2}(\pi r^2)=\frac{\omega q r^3 dr}{R^2}$$

$$\mu = \int d\mu = \frac{\omega q}{R^2} \int_0^R r^3 dr = \frac{\omega q R^2}{4}$$

34.26. For a square of side a,

$$4a = l \qquad \ldots(1)$$

By Problem 34.15 the induction at the center of the square is

$$B_s = \frac{2\sqrt{2}\,\mu_0 i}{\pi a} = \frac{8\sqrt{2}\,\mu_0 i}{\pi l} \qquad \ldots(2)$$

where use has been made of (1).

For a circle of radius R, ...(3)

$$2\pi R = l$$

The induction at the center of the circle is

$$B_c = \frac{\mu_0 i}{2R} = \pi \frac{\mu_0 i}{l} \qquad \ldots(4)$$

where use has been made of (3).

Dividing (2) by (3),

$$\frac{B_s}{B_c} = \frac{8\sqrt{2}}{\pi^2} = 1.15 \qquad \ldots(5)$$

As the right hand side is greater than unity, we conclude that $B_s > B_c$, i.e. the square yields a larger value for the magnetic induction at the center than the circle.

34.27. (*a*) The magnetic induction set up by the large loop at its center is

$$B = \frac{\mu_0 i}{2R} = \frac{(4\pi \times 10^{-7}\ \text{weber/amp-m})(15\ \text{amp})}{(2)(0.1\ \text{meter})}$$

$$= 9.4 \times 10^{-5}\ \text{weber/meter}^2$$

(*b*) Torque, $\tau = \mu B \sin\theta = \mu B$, as $\theta = 90°$.

as $\theta = 90°$

Here, $\mu = NiA = (50)(1.0\ \text{amp})\pi(0.01\ \text{meter})^2$

$= 1.57 \times 10^{-2}\ \text{amp-meter}^2$

$\therefore$ $\tau = (1.57 \times 10^{-2}\ \text{amp-meter}^2)(9.4 \times 10^{-5}\ \text{weber/meter}^2)$

$= 1.48 \times 10^{-6}$ nt-meter.

34.28. Applying Ampere's theorem to the rectangular loop *abcd* as in Fig. 34.28.

$$\oint \mathbf{B}.d\mathbf{l} = \mu_0\, i = 0$$

as no current flows. Further, assume that the magnetic field drops to zero along the side *cd*.

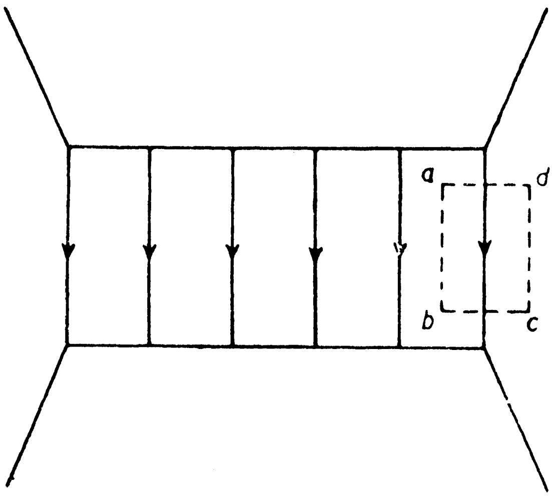


Fig. 34.28

Now, the contribution of the path *bc* and *da* is zero to the integral as B is perpendicular to *ad* and *bc*. Also, the path *cd* does not contribute anything as by our assumption B is zero along *cd*. The only contribution to the integral then comes from the path *ab*. If the length *ab* is L, then

$$BL=0$$

or $$B=0$$

along *ab*, which is absurd. We, therefore, conclude that our assumption that the field along *cd* is zero is wrong.

SUPPLEMENTARY PROBLEMS

S.34.1. The wires labeled 1, 3, 6, and 8 above are within the closed path. According to Ampere's law,

$$\oint \mathbf{B}.d\mathbf{l}=\mu_0 i \qquad \ldots(1)$$

Here, the direction of traverse around the loop is clockwise when one faces the loop, and the current is considered positive if its direction is away from the observer.

As in the present case the direction of traverse around the loop is counter-clockwise the integral yields a negative sign for the right hand side of (1) with the current into the plane of paper as positive.

$$i=ki_0\,;\ (k=-1,\ -3,\ +6 \text{ and } +8)$$

By Problem,

$\therefore \quad i = -i_0 - 3i_0 + 6i_0 + 8i_0 = 10i_0$

Thus, $\oint \mathbf{B}.d\mathbf{l} = -10\mu_0 i_0$.

S.34.2. One must consider the effect of the magnetic field due to the current i_1 (Fig. S.34.2) on segments like A and C on the conductor carrying the current i_2. The net result is that the torque tends to align the conductors with the currents running parallel. When this state of affairs is reached the conductors with parallel currents will begin to attract each other.

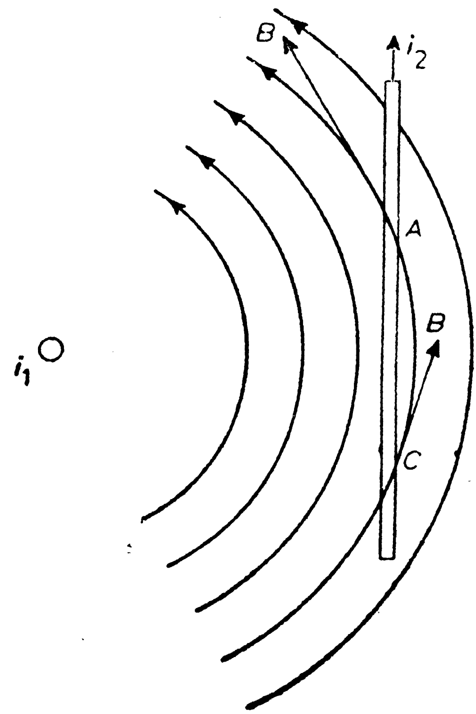


Fig. S.34.2

S.34.3. Consider the force/meter on wire labeled 1 by 2, 3 and 4. As the currents in the wires are equal, $i_1 = i_2 = i_3 = i_4 = i$. The force of 2 on 1 per meter is given by

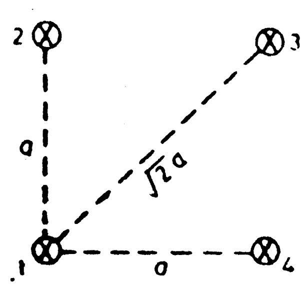


Fig. S.34.3

$$F_{21} = \frac{\mu_0 i_1 i_2}{2\pi a} = \frac{\mu_0 i^2}{2\pi a}$$

acting along the line joining 1 and 2.

Similarly, the force of 4 on 1 per meter is given by

$$F_{41} = \frac{\mu_0 i^2}{2\pi a}$$

acting along the line joining 1 and 4.

As the magnitude of F_{21} and F_{41} are equal and as they act at right angles their resultant, which is given by $\sqrt{F_{21}^2+F_{41}^2}=\sqrt{2}\,F_{21}$, would lie along the line joining 1 and 3.

Now, the force of 3 on 1 is given by

$$F_{31}=\frac{\mu_0 i^2}{2\pi(\sqrt{2}\,a)}$$

along the line joining 1 and 3.

Hence, the resultant is given by

$$F=\sqrt{2}\,F_{21}+F_{31}=\frac{\sqrt{2}\mu_0 i^2}{2\pi a}+\frac{\mu_0 i^2}{2\pi\sqrt{2}a}$$

$$=\frac{\mu_0 i^2}{2\pi a}\left(\sqrt{2}+\frac{1}{\sqrt{2}}\right)=\frac{3\mu_0 i^2}{2\sqrt{2}\pi a}=\frac{3\sqrt{2}\mu_0 i^2}{4\pi a}$$

along the line joining 1 and 3, i.e. towards the center of the square.

S.34.4. The magnetic field at P, the center of the hole, is obtained by considering it to arise due to two current densities (Fig. S.34.4),

(i) a current density, $j=i/\pi\,(R^2-a^2)$ carried by the cylinder of radius R, and

(ii) a current density $-j$ carried by a cylinder of radius a. The resultant magnetic induction is the vector sum of the effects under (i) and (ii). The contribution due to (i) is obtained by applying Ampere's law.

$$\oint \mathbf{B}.d\mathbf{l}=\mu_0 i'=\mu_0 i\frac{\pi b^2}{\pi(R^2-a^2)}=\frac{\mu_0 i b^2}{(R^2-a^2)}$$

$$(B)(2\pi b)=\frac{\mu_0 i b^2}{(R^2-a^2)}$$

or $$B=\frac{\mu_0 i b}{2\pi(R^2-a^2)}$$

The induction at P due to equivalent solid conductor of radius a carrying an assigned current opposite to the above is zero as no current will be contained within the path of zero radius.

The resultant magnetic induction at P, which is given by the summation of the two foregoing factors is then

$$B=\frac{\mu_0 i b}{2\pi(R^2-a^2)}$$

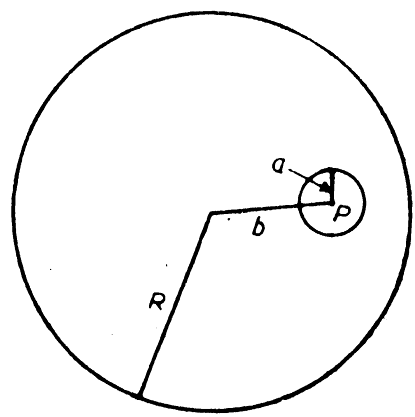


Fig. S.34.4

S.34.5. The resultant B at the center C of the circular loop may be considered as the superposition of fields B_1 and B_2 in the upper and lower semi-circles respectively. Now, the current i divides itself equally in the two semi-circular paths. By Problem 34.18, the contribution to the induction from the upper semi-circle with the current running clockwise would be

$$B_1=\frac{\mu_0}{4R}\left(\frac{1}{2}i\right)=\frac{\mu_0 i}{8R}$$

into the plane of this figure at right angles to the page. An equal contribution to the induction is made from the lower semi-circle viz.,

$$B_2=\frac{\mu_0 i}{8R}$$

But since the current is running in the counter-clockwise sense, B_2 will be pointing out of the page at right angles to the plane of the figure. Thus, the resultant induction $\mathbf{B}=\mathbf{B}_1+\mathbf{B}_2=0$.

S.34.6. (*a*) According to the Biot-Savart law, dB is given in magnitude by

$$Bd=\frac{\mu_0 i}{4\pi}\;\frac{dx \sin\theta}{r^2}$$

As the points S and P lie on the axis of the current $\theta=0$ for all the elements of the wire and consequently, $B_P=B_S=0$. Let us calculate the induction at Q. From Fig. S.34.6 (*a*),

$$\sin\theta=\sin(\pi-\theta)=\frac{L}{r}=\frac{L}{\sqrt{x^2+L^2}}$$

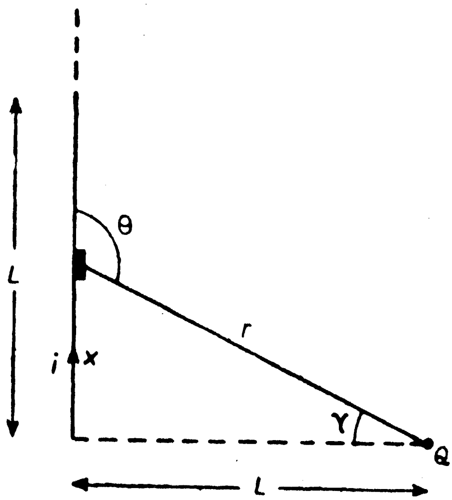


Fig. S.34.6 (a)

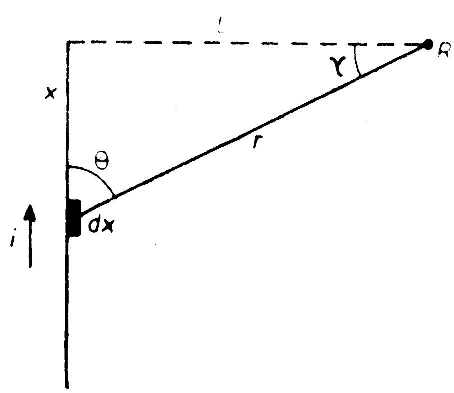


Fig. 34.6 (b)

$$B_Q = \frac{\mu_0 i}{4\pi} \int_0^L \frac{L\,dx}{(x^2+L^2)^{3/2}}$$

Set $\quad x = L \tan\gamma$

$\quad\quad dx = L \sec^2\gamma\, d\gamma$

Then, $\quad B_Q = \frac{\mu_0 i}{4\pi} \int_0^{45^\circ} \frac{L^2 \sec^2\gamma\, d\gamma}{L^3 \sec^3\gamma} = \frac{\mu_0 i}{4\pi L} \int_0^{45^\circ} \cos\gamma\, d\gamma$

$$= \frac{\mu_0 i}{4\pi L} \sin\gamma \Big|_0^{45^\circ} = \frac{\mu_0 i}{4\sqrt{2}\pi L} = \frac{\mu_0 i \sqrt{2}}{8\pi L}$$

into the plane of paper.

Finally we calculate B_R. From Fig. 34.6 (*b*),

$$dB = \frac{\mu_0 i}{4\pi} \frac{dx \sin\theta}{r^2}$$

$$\sin\theta = \frac{L}{r} = \frac{L}{\sqrt{x^2+L^2}}$$

$$B_R = \frac{\mu_0 i}{4\pi} \int \frac{L\,dx}{(x^2+L^2)^{3/2}}$$

Set $\quad x = L \tan\gamma$

$\quad\quad dx = L \sec^2\gamma\, d\gamma$

Then, $\quad B_R = \frac{\mu_0 i}{4\pi} \int_0^{45^\circ} \frac{L^2 \sec^2\gamma\, d\gamma}{L^3 \sec^3\gamma}$

$$= \frac{\mu_0 i}{4\pi L} \int_0^{45^\circ} \cos\gamma\, d\gamma = \frac{\mu_0 i}{4\pi L} \sin\gamma \Big|_0^{45^\circ}$$

$$= \frac{\mu_0 i}{4\sqrt{2}\pi L} = \frac{\mu_0 i \sqrt{2}}{8\pi L}$$

into the plane of figure.

(*b*) Calling the induction at T due to the sides labeled 1, 2, ...6 of the closed loop, by B_1, B_2 ...B_6, we have

$$B_1 = B_4 = 0$$

$$B_2 = B_3 = \frac{\mu_0 i \sqrt{2}}{8\pi a}$$

$$B_5 = B_6 = -\frac{\mu_0 i\sqrt{2}}{16\pi a}$$

$$\therefore \quad B_T = B_1 + B_2 + B_3 + B_4 + B_5 + B_6$$

$$= \frac{2(\mu_0 i\sqrt{2})}{8\pi a} - \frac{2(\mu_0 i\sqrt{2})}{16\pi a} = \frac{\mu_0 i\sqrt{2}}{8\pi a}$$

in the plane of figure.

S.34.7. Using the result of Problem 34.18, the induction at C due to the inner semi-circle of radius R_1 is, $B_1 = \mu_0 i/4R_1$ (current being clockwise) and that due to outer semi-circle of radius R_2 is

$$B_2 = \frac{\mu_0 i}{4R}, \text{ (current being counter-clockwlse)}$$

The straight sectors AH and JD which upon extension pass through C give zero induction. Therefore, the resultant induction at C is,

$$B = B_1 + B_2 = \frac{\mu_0 i}{4R_1} - \frac{\mu_0 i}{4R_2} = \frac{\mu_0 i}{4}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

into the page.

S.34.8. Due to inner arc, the induction at C is

$$B_1 = \int dB_1 = \frac{\mu_0 i \sin\alpha}{4\pi R^2_1} \int dx$$

where $\quad \alpha = 90°$. Also $dx = R_1\, d\theta$.

$$\therefore \quad B_1 = \frac{\mu_0 i\,\theta}{4\pi R_1}$$

Similarly, due to outer arc,

$$B_2 = -\frac{\mu_0 i\theta}{4\pi R_2}$$

the negative sign arises due to the fact that the current has reversed its direction. As the radial part of the path points towards C, it does not contribute to the B. Therefore, the resultant induction is

$$B = B_1 + B_2 = \frac{\mu_0 i\theta}{4\pi}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

S.34.9. (*a*) The resultant B at the center is given by the superposition of B_s due to the straight conductor and B_c due to the circular path, both of them being directed out of the plane of figure.

$$B_s = \frac{\mu_0 i}{2\pi R}$$

$$B_c = \frac{\mu_0 i}{2R}$$

$$\therefore \qquad B = B_s + B_c$$

$$= \frac{\mu_0 i}{2\pi R} + \frac{\mu_0 i}{2R} = \frac{\mu_0 i}{2R}\left(1 + \frac{1}{\pi}\right)$$

out of the page.

(*b*) As before B_s points up out of the paper. But, now B_c lies in the plane of paper being in the direction of the current in the straight conductor, its magnitude remaining the same as before. At *C*, the resultant induction is given by

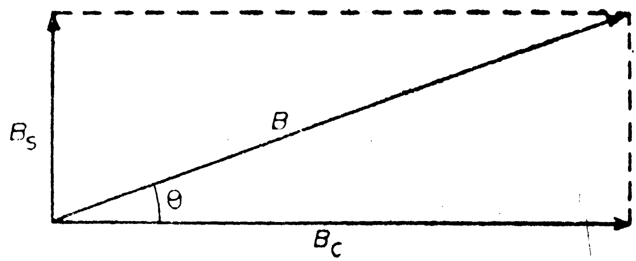


Fig. S.34,9

$$B = \sqrt{B_s^2 + B_c^2} = \sqrt{\left(\frac{\mu_0 i}{2\pi R}\right)^2 + \left(\frac{\mu_0 i}{2R}\right)^2}$$

$$= \frac{\mu_0 i}{2\pi R}\sqrt{1+\pi^2}$$

B is inclined at an angle θ, given by $\theta = \tan^{-1}\frac{B_s}{B_c} = \tan^{-1}\frac{1}{\pi} = 18°$,

out of the page.

35 FARADAY'S LAW

35.1. The induced emf, $E=-N\frac{d\phi_B}{dt}$

where N is the number of turns, and $\frac{d\phi_B}{dt}$ is the rate of change of flux.

Now, $\phi_B=BA$

where A is the area of cross-section.

Change in flux $\Delta\phi_B$ for each turn of the coil is

$$\Delta\phi_B=A\Delta B=(0.001 \text{ meter}^2)(2 \text{ weber/meter}^2)$$
$$=2\times10^{-3} \text{ weber.}$$

since the magnetic induction changes from 1.0 weber/meter^2 to -1.0 weber/meter^2.

The current i is given by

$$i=\frac{E}{R}=-\frac{N}{R}\frac{\Delta\phi_B}{\Delta t}=\frac{\Delta q}{\Delta t}$$

$\therefore$ The quantity of charge Δq flowing through the circuit is

$$\Delta q=\frac{N}{R}\Delta\phi_B=\frac{(100)(2\times10^{-3} \text{ weber})}{(10 \text{ ohm})}$$
$$=2\times10^{-2} \text{ coul.}$$

35.2. The long solenoid of Example 1 has 200 turns/cm and carries a current of 1.5 amp; its diameter is 3.0 cm. The current in the solenoid is reduced to zero and then raised to 1.5 amp in the other direction at a steady rate over 0.05 sec.

Cross-section area of the solenoid,

$$A=\frac{\pi d^2}{4}=\frac{\pi}{4}(0.03 \text{ meter})^2$$
$$=7\times10^{-4} \text{ meter}^2$$

Field B in the solenoid

$$\mu_0 ni=(4\pi\times10^{-7} \text{ weber/amp-m})(200\times100/\text{meter})(1.5 \text{ amp})$$
$$=3.8\times10^{-2} \text{ weber/meter}^2.$$

$$\phi_B = BA = (3.8\times10^{-2}\text{ weber/meter}^2)(7\times10^{-4}\text{ meter}^2)$$
$$=2.66\times10^{-5}\text{ weber.}$$
$$\Delta\phi_B = 2\times2.66\times10^{-5}\text{ weber} = 5.32\times10^{-5}\text{ weber}$$

Rate change of flux, $\dfrac{\Delta\phi_B}{\Delta t} = E = \dfrac{5.32\times10^{-5}\text{ weber}}{0.05\text{ sec}}$

$$=1.06\times10^{-3}\text{ weber/sec.}$$

The current in the coil is

$$i=\frac{E}{R}=\frac{Nd\phi_B}{Rdt}=\frac{100}{(5\text{ ohm})}(1.06\times10^{-3}\text{ weber/sec})$$
$$=2.1\times10^{-2}\text{ amps.}$$

35.3. As the loop is wo ed, it normal rotates about the field direction at a constant angle of 30°. In this process the lines of force cutting the loop would not vary. Hence, in accordance with Faraday's law no emf will be produced in the loop.

35.4. Area of cross-section of the loop,

$$A=\frac{\pi}{4}D^2=\frac{\pi}{4}(0.1\text{ meter})^2$$
$$=7.85\times10^{-3}\text{ meter}^3$$

Length of wire, $l=\pi D=\pi(0.1\text{ meter})=0.314$ meter

Area of cross-section of wire,

$$a=\frac{\pi}{4}d^2=\frac{\pi}{4}(0.1\times2.54\times10^{-2}\text{m})^2$$
$$=5\times10^{-6}\text{ meter}^2.$$

Resistance of the wire, $R=\rho l/a$

$$=\frac{(1.7\times10^{-8}\text{ ohm-meter})(0.314\text{ meter})}{(5\times10^{-6}\text{ meter}^2)}=1.05\times10^{-3}\text{ohm.}$$

$$i=\frac{E}{R}=\frac{A}{R}\frac{dB}{dt}$$

or $$\frac{dB}{dt}=\frac{iR}{A}=\frac{(10\text{ amp})(1.05\times10^{-3}\text{ ohm})}{(7.85\times10^{-3}\text{ meter}^2)}$$
$$=1.3\text{ weber/meter}^2\text{-sec.}$$

35.5. Mass, $m=al\,d_0$

$$=(\pi r^2)(2\pi R)\,d_0 \qquad \ldots(1)$$

where d_0 is the density of copper, l the length of the wire and $a=\pi r^2$, the cross-section area of the wire. If A is the area of the loop and R_0 the resistance then the induced current in the loop is

$$i=\frac{A}{R_0}\frac{dB}{dt}=\frac{\pi R^2\,dB}{R_0\,dt} \qquad \ldots(2)$$

But $$R_0=\frac{\rho l}{a}=\frac{\rho(2\pi R)}{\pi r^2}=\frac{2R\rho}{r^2} \qquad ...(3)$$

Use (3) in (2) to eliminate R_0,

$$i=\frac{\pi R r^2}{2\rho}\frac{dB}{dt} \qquad ...(4)$$

From (1) we have,

$$\pi R r^2=\frac{m}{2\pi d_0} \qquad ...(5)$$

Use (5) in (4) to find

$$i=\frac{m}{4\pi\rho d_0}\frac{dB}{dt} \qquad ...(6)$$

It is clear from (6) that i is independent of l and r (size of the wire) and R (size of the loop).

35.6. Radius of the wire, $r=0.02 \text{ in.}=0.02\times2.54\times10^{-2}$ meter
$=5.08\times10^{-4}$ meter.

Area of cross-section of wire, $a=\pi r^2=\pi(5.08\times10^{-4}\text{ meter})^2$
$=81\times10^{-8}\text{ meter}^2$.

Resistance of the wire, $R_0=\frac{\rho l}{a}=\frac{(1.7\times10^{-8}\text{ ohm-m})(0.5\text{ meter})}{(81\times10^{-8}\text{ meter}^2)}$
$=1.05\times10^{-2}$ ohm.

Radius of the loop, $R=\frac{l}{2\pi}=\frac{0.5\text{ meter}}{2\pi}=7.96\times10^{-2}$ meter.

Area of the loop, $A=\pi R^2=\pi(7.96\times10^{-2}\text{ meter})^2$
$=0.02\text{ meter}^2$

$dB/dt=100$ gauss/sec$=10^{-2}$ weber/meter^2-sec.

Induced emf, $E=A\frac{dB}{dt}=(0.02\text{ meter}^2)(10^{-2}\text{ weber/meter}^2\text{-sec})$

$=2\times10^{-4}$ volt.

Joule heating, $P_J=i^2R_0=\frac{E^2}{R_0}=\frac{(2\times10^{-4}\text{ volt})^2}{1.05\times10^{-2}\text{ ohm}}$

$=3.8\times10^{-6}$ watts.

35.7. As the north pole enters the coil the direction of current in the face S_1 of the coil is counter-clockwise, Fig. 35.7 (*a*). The current rises to maximum when the magnet is half way through, and as the magnet continues to move in the same direction the

current in the coil decreases but continues to flow in the counter-clockwise sense as viewed along the path of the magnet. Fig. 35.7 (*b*) shows qualitatively the plot of i the current as a function of x, the distance of the outer of magnet from the center of the loop.

Joule heating is given by $P_J = i^2R$. Fig. 35.7 (*c*) shows the qualitative plot of P_J as a function of x.

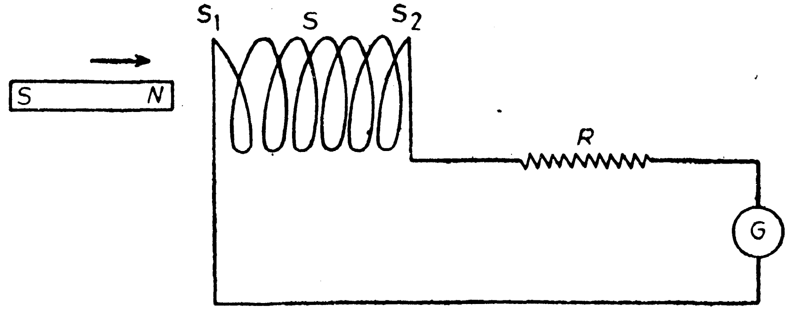


Fig. 35.7 (a)

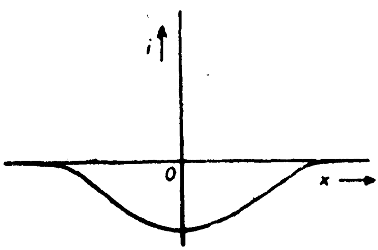


Fig. 35.7 (b)

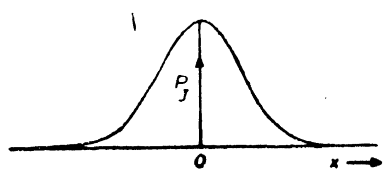


Fig. 35.7 (c)

35.8. (*a*) The magnitude of the emf induced in the loop may be computed from the rate of change of flux through the loop.

Let the plane of the loop make an angle α with the normal to the field. Then the flux through the loop is

$$\phi = AB \cos \alpha \qquad \ldots(1)$$

where A is the area of the loop. The rate of change of flux is then

$$\frac{d\phi}{dt} = -AB \sin \alpha \frac{d\alpha}{dt} \qquad \ldots(2)$$

The induced emf is

$$E = -N\frac{d\phi}{dt} = NAB\omega \sin \alpha \qquad \ldots(3)$$

where $\omega = d\alpha/dt$ is the angular velocity of the loop and N is the number of turns in the loop.

Put, $\omega=2\pi\nu$

$A=ab$

$\alpha=\omega t$

Eq. (3) becomes

$$E=2\pi\nu\ Nba\ B \sin 2\pi\nu t=E_0 \sin 2\pi\nu t \qquad \ldots(4$$

where $E_0=2\pi\nu\ NbaB$ is the maximum value of the induced emf.

(*b*) From (4) we have,

$$NA=\frac{E_0}{2\pi\nu B}=\frac{150 \text{ volt}}{(2\pi)(60 \text{ rev/sec})(0.5 \text{ weber/meter})^2}$$

$$=5/2\pi \text{ turn-meter}^2$$

35.9. By Problem 35.8 we have

$$E=2\pi\nu NbaB \sin 2\pi\nu t=E_0 \sin 2\pi\nu t.$$

The amplitude of the induced voltage is

$$E_0=2\pi\nu Nba\ B=2\pi\nu NAB$$

where we have set $ba=A$, the area of the loop.

Putting $A=\pi R^2/2$, the area of semi-circular loop, and $N=1$, we have

$$E_0=\pi^2\nu R^2 B$$

Amplitude of induced current is

$$i_0=\frac{E_0}{R_M}=\frac{\pi^2\nu R^2 B}{R_M}$$

35.10. The emf developed between the axis of the disk and its rim is

$$E=\frac{B\omega R^2}{2}$$

$$=\tfrac{1}{2}(1.0 \text{ weber/meter}^2)(2\pi\times 30 \text{ rev/sec})(0.05 \text{ meter})^2$$

$$=0.24 \text{ volts.}$$

35.11. $E=\bar{B}lv=\bar{B}v\ (b-a)$

Now, $B(r)=\dfrac{\mu_0 i}{2\pi r}$

$$\bar{B}=\frac{1}{(b-a)}\int_a^b \bar{B}(r)\,dr=\frac{\mu_0 i}{2\pi\,(b-a)}\int_a^b \frac{dr}{r}=\frac{\mu_0 i}{2\pi\,(b-a)}\ln\frac{b}{a}$$

$$\therefore \qquad E = \frac{\mu_0 i v}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$= \frac{(4\pi\times 10^{-7}\ \text{weber/amp·m})(100\ \text{amp})(5\ \text{meter/sec})}{2\pi} \ln\left(\frac{20\ \text{cm}}{1\ \text{cm}}\right)$$

$$= 3\times 10^{-4}\ \text{volt.}$$

35.12. (*a*) Force acting on the wire is

$$F = i\mathbf{d}\times\mathbf{B} = idB$$

$\therefore$ Acceleration, $a = \dfrac{F}{m} = \dfrac{idB}{m}$

At time t, velocity is given by

$$v = 0 + at = \frac{idBt}{m}$$

The direction is from right to left.

(*b*) By Faraday's law

$$E = -\frac{d\phi_B}{dt} = Bdv$$

$$\therefore \qquad v = \frac{E}{Bd}$$

(*c*) The terminal speed of the wire is $v_T = E/Bd$. The wire is being resisted by a constant force F so that it is moving with constant speed v.

Now, $\quad F = Bid \qquad \ldots(1)$

and the rate of working is

$$P = Fv_T = Bidv_T \qquad \ldots(2)$$

But the induced emf is equal to the rate of change of flux through the circuit enclosed by the moving wire and the two rails.

$$E = B\frac{dA}{dt} = Bdv_T \qquad \ldots(3)$$

where A is the area of the circuit. In order to provide current against the induced emf defined by (3), the battery must work at the rate

$$Ei = Bdv_T i = P \qquad \ldots(4)$$

where use has been made of (2). Thus, the entire work is done by the battery to slow down the wire to constant speed (terminal speed), the induced emf getting completely cancelled by that provided by the battery so that the net current in the circuit is zero.

35.13. (*a*) $\frac{d\phi_B}{dt}=12t+7$

$$|E|=\left.\frac{d\phi_B}{dt}\right|_{t=2}=(12\times2+7) \text{ milliweber/sec}$$

$$=31 \text{ millivolt.}$$

(*b*) Direction of current through R is from left to right.

35.14. (*a*) The emf induced in the rod is

$$E=Blv=(1.0 \text{ weber/meter}^2)(0.5 \text{ meter})(8 \text{ meter/sec})$$
$$=4.0 \text{ volts.}$$

From Lenz's law E must be counter-clockwise.

(*b*) Force required to keep the rod in motion is

$$F=\frac{B^2l^2v}{R}=\frac{(1.0 \text{ weber/meter}^2)^2(0.5 \text{ meter})^2 (8 \text{ meter/sec})}{(0.4 \text{ ohm})}$$
$$=5.0 \text{ nt.}$$

(*c*) Rate at which mechanical work is done by the force F is

$$P=Fv=\left(\frac{B^2l^2v}{R}\right)v=\frac{B^2l^2v^2}{R}$$

Joule heating is

$$P_J=\frac{B^2l^2v^2}{R}$$

$\therefore \qquad P=P_J$

Thus, mechanical power=electrical power.

$$P=Fv=(5.0 \text{ nt})(8 \text{ meter/sec})=40 \text{ watts.}$$

35.15. (*a*) Gravitational force acting on the wire down the rail,

$$F_g=mg \sin\theta \qquad \text{...(1)}$$

The component of magnetic induction normal to the plane of rails is $B\cos\theta$. The magnetic force on the wire up the rail is

$$F_M=\frac{B^2\cos^2\theta l^2v}{R} \qquad \text{...(2)}$$

For steady state velocity, the net force must be zero. That is

$$F_M=F_g$$

or, $\frac{B^2\cos^2\theta\, l^2v}{R}=mg\sin\theta$

whence, $$V=\frac{mgR\sin\theta}{B^2l^2\cos^2\theta} \qquad \ldots(3)$$

(*b*) Gravitation does work at the steady rate of

$$P=F_g v=mgv\sin\theta \qquad \ldots(4)$$

where use has been made of (1).

Joule heating is given by

$$P_J=\frac{B^2\cos^2\theta l^2v^2}{R}=mgv\sin\theta \qquad \ldots(5)$$

where use has been made of (3). Since (4) and (5) are identical we conclude that Joule heat appears in the resistor at the same rate as that done by gravitational force—a result which is consistant with the conservation of energy.

(*c*) If B were directed down, then the force due to magnetic induction would have a component down the plane of rails and would reinforce the component of gravitational force and the net force would be

$$F_{(net)}=\frac{B^2\cos^2\theta l^2v}{R}+mg\sin\theta$$

The wire would therefore, suffer acceleration, acquiring ever increasing speed, this being the case of non-uniform acceleration.

35.16. The induced emf is

$$E=-\frac{Nd\phi_B}{dt}$$

$$i=\frac{dq}{dt}=\frac{E}{R}=-\frac{N}{R}\frac{d\phi_B}{dt}$$

$$dq=-\frac{N}{R}d\phi_B$$

$$\int dq=q=-\frac{N}{R}\int_{\phi_2}^{\phi_1}d\phi_B=\frac{N}{R}(\phi_2-\phi_1)$$

35.17. Magnitude of emf developed is

$$E=\frac{d\phi_B}{dt}=A\frac{dB}{dt}=\pi r^2\frac{dB}{dt}$$

Electric field E_E at a distance r from the center is

$$E_E=\frac{E}{2\pi r}=\frac{\pi r^2}{2\pi r}\frac{dB}{dt}=\frac{1}{2}r\frac{dB}{dt}$$

Force on electron is,

$$F=E_E e=\frac{e}{2}r\frac{dB}{dt}$$

Acceleration is,

$$a=\frac{F}{m}=\frac{1}{2}\frac{e}{m}r\frac{dB}{dt}$$

At the point a, the electron experiences an acceleration,

$$a=\frac{1}{2}\frac{(1.6\times10^{-19}\text{ coul})}{(9.1\times10^{-31}\text{ kg})}(0.05\text{ meter})(0.01\text{ weber/meter}^2\text{-sec})$$

$$=4.4\times10^{7}\text{ meter/sec}^2\text{, to the right.}$$

At the point b, the electron has acceleration, $a=0$

At the point c, the electron has acceleration,

$$a=4.4\times10^{7}\text{ meter/sec}^2\text{, to the left}$$

35.18. The electric field E_E at any point b on the rod at distanee r from the center is perpendicular to r as in Fig. 35.18. The magnitude of E_E is given by

$$E_E=\frac{1}{2}r\frac{dB}{dt} \qquad \ldots(1)$$

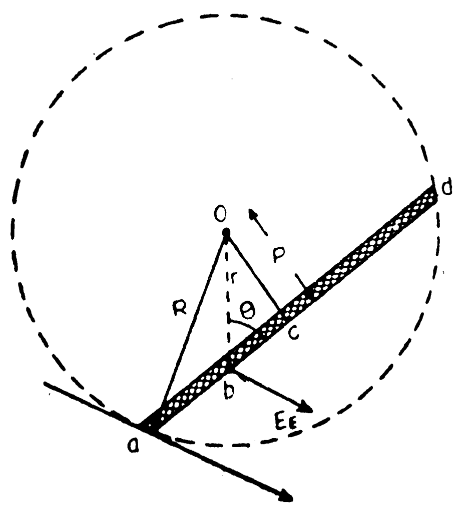


Fig. 35.18

Resolve E_E along two mutually perpendicular directions, $E_{E\parallel}$ along the length of the rod and $E_{E\perp}$, perpendicular to it.

$$E_{E\parallel}=E_E \sin\theta=(\tfrac{1}{2}\ r\,dB/dt)(p/r)$$
$$=\tfrac{1}{2}\,p\ dB/dt$$

where $\quad p=Oc=\sqrt{R^2-l^2/4}$

According to Faraday's law

$$\mathcal{E}=\int \mathbf{E}_E.d\mathbf{l}=\int (\mathbf{E}_{E\perp}+\mathbf{E}_{E\parallel}).d\mathbf{l})$$

But $\quad \int \mathbf{E}_{E\perp}.d\mathbf{l}=0$

$$\therefore \quad \mathcal{E}=\int \mathbf{E}_{E\parallel}.d\mathbf{l}=\int_0^l \frac{p}{2}\,\frac{dB}{dt}\,dt$$

$$=\frac{dB}{dt}\,\frac{l}{2}\sqrt{R^2-\left(\frac{l}{2}\right)^2}$$

35.19. Fig 35.19 shows the plot of $B\,(r)$ against r.

Area under the curve is

$$A=(3420)(200\times 1)\text{ gauss-cm}$$
$$=6.84\times 10^5\text{ gauss-cm}$$

$$\bar{B}=\frac{A}{R}=\frac{6.84\times 10^5\text{ gauss-cm}}{84\text{ cm}}=8143\text{ gauss}$$

Now, from the graph we note that at $r=R=84$ cm,

$$B_R=4000\text{ gauss.}$$

$$\therefore \quad 2B_R=8000\text{ gauss}$$

Thus, the relation $\bar{B}=2B_R$ is nearly satisfied.

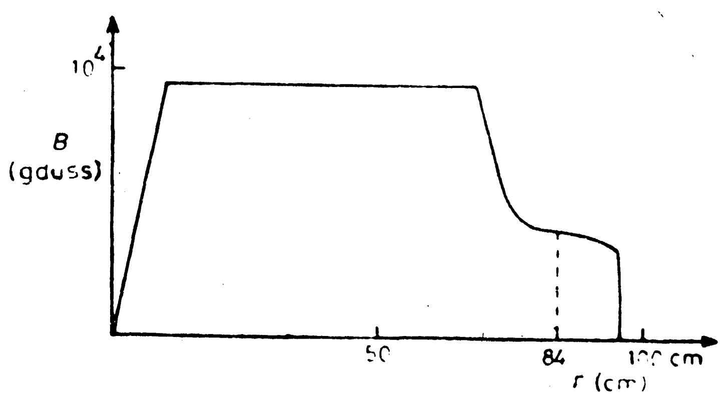


Fig. 35.19

35.20. Apply Faraday's law to the rectangular path *abcda* in Fig. 35.20. Then

$$E=\oint \mathbf{E}.d\mathbf{l}=0 \qquad ...(1)$$

as emf is absent. Now the contribution to the integral from the horizontal paths *bc* and *da* is zero as E is perpendicular to these paths. If we now suppose that along *cd* the electric field is zero, then (1) gives,

$$EL=0$$

where $ab=L$. Thus $E=0$ along *ab*, which is contrary to the problem. Hence, our assumption that E drops to zero abruptly outside the parallel plate is wrong.

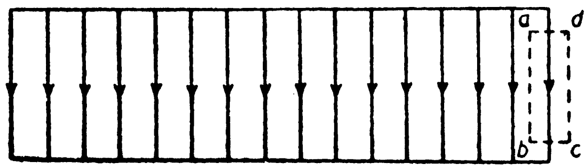


Fig. 35.20

SUPPLEMENTARY PROBLEMS

S.35.1. [emf]=[electric field][distance]

$$=[\text{force/charge}][\text{distance}]$$
$$=[MLT^{-2}Q^{-1}][L]=[ML^2T^{-2}Q^{-1}]$$

$$[d\phi_B/dt]=[\phi_B][T^{-1}]=[B][\text{area}][T^{-1}]$$
$$=[\text{force/(velocity)(charge)}][L^2][T^{-1}]$$
$$=[MLT^{-2}/LT^{-1}Q][L^2][T^{-1}]$$
$$=[ML^2T^{-2}Q^{-1}]$$

Thus, $[\text{emf}]=[d\phi_B/dt]$.

S.35.2. (*a*) The resistance of the circuit *ADCB* can be rendered approximately constant by choosing the resistance of *AB* equal to 1.2×10^{-5} ohm and negligible resistance for the section *ADCB*.

Induced emf, $E=-d\phi_B/dt=Bvl$

$$=(6\times10^{-6}\text{ weber/m}^2)(0.5\text{ meter/sec})(2.0\text{ meter})$$
$$=6\times10^{-6}\text{ volt.}$$

(*b*) Electric field, $E_E=\frac{E}{l}=\frac{6\times10^{-5}\text{ volt}}{2\text{ meter}}=3\times10^{-5}$ volt/meter

(*c*) Force on electron, $F=E_Le$

$$=(3\times10^{-5}\text{ volt/meter})(1.6\times10^{-19}\text{ coul})$$
$$=4.8\times10^{-24}\text{ nt.}$$

(*d*) Current, $i=\frac{E}{R}=\frac{6\times10^{-5}\text{ volt}}{1.2\times10^{-5}\text{ ohm}}=5$ amp

(*e*) The force due to the induced emf must be counter balanced by an equal and opposite force.

$$F=ilB=(5\text{ amp})(2\text{ meter})(6\times10^{-5}\text{ weber/m}^2)$$
$$=6\times10^{-4}\text{ nt.}$$

(*f*) Rate of work, $P=Fv=(6\times10^{-4}\text{ nt})(0.5\text{ meter/sec})$

$$=3\times10^{-4}\text{ watts.}$$

(*g*) Rate of joule heating, $P_J=i^2R=(5\text{ amp})^2(1.2\times10^{-5}\text{ ohm})$

$$=3\times10^{-4}\text{ watts.}$$

S.35.3. (*a*) The induction along the axis of a circular current loop of radius R carrying current i at large distance $x(x \gg R)$, is given by

$$B=\frac{\mu_0 iR^2}{2x^3}$$

The magnetic flux, $\phi_B=BA=B\pi r^2$

where A is the area of the smaller loop.

$$\phi_B=\frac{\mu_0 i\pi R^2r^2}{2x^3} \qquad \ldots(1)$$

(*b*)
$$E=-\frac{d\phi_B}{dt}=-\frac{d\phi_B}{dx}\frac{dx}{dt}=-\frac{d\phi_B}{dx}v \qquad \ldots(2)$$

Differentiating (1) with respect to x and setting $x=NR$ in the resulting expression,

$$E=-\frac{\mu_0 i\pi R^2r^2v}{2}\frac{d}{dx}\left(\frac{1}{x^3}\right)$$

$$=\frac{3}{2}\mu_0 i\pi R^2r^2v\left.\frac{1}{x^4}\right|_{x=NR}$$

$$=\frac{3\mu_0 i\pi r^2v}{2N^4R^2} \qquad \ldots(3)$$

(*c*)
$$E=-\frac{d\phi_B}{dx}v \qquad \ldots(4)$$

Since v is positive, expression (3) shows that E is positive and (4) showns that $d\phi_B/dx$ is negative i.e. the flux through the smaller loop is decreasing. The direction of current in the smaller loop will be such as to oppose the decrease in flux, i.e. the direction of the current will be in the same sense as in the larger loop.

S.35.4. (*a*) The projected area in a plane normal to B is.

$$A=\pi r^2 \cos 45°$$

$$=\frac{\pi}{\sqrt{2}}(0.037 \text{ meter})^2=3.04\times 10^{-3} \text{ meter}^2$$

The induced emf is

$$E=-\frac{d\phi_B}{dt}=-A\frac{dB}{dt}$$

$$=-(3.04\times 10^{-3} \text{ meter}^2)\left(-\frac{76\times 10^{-3} \text{ weber/m}^2}{4.5\times 10^{-3} \text{ sec}}\right)$$

$$=5.13\times 10^{-2} \text{ volt.}$$

(*b*) The emf produced in each of the two semi-circular loops would be equal in magnitude but in the opposite sense. The net emf in the complete loop would be zero.

S.35.5. (*a*) The induced emf is given by

$$E=-\frac{d\phi_B}{dt}$$

The current through the loop of wire is

$$i=\frac{E}{R}=-\frac{1}{R}\frac{d\phi_B}{dt} \quad \text{...(1)}$$

But, $$i=\frac{dq}{dt}. \quad \text{...(2)}$$

Comparing (1) and (2),

$$dq=-1/R\, d\phi_B$$

Integrating,

$$q=\int_{t_1} dq=-\frac{1}{R}\int_{t_1}^{t_2} d\phi_B=-\frac{1}{R}\Big[\phi_B(t_2)-\phi_B(t_1)\Big]$$

The result is independent of the manner in which B is changing.

(*b*) Since the above result is independent of B, it is possible that B was changing in the time interval t_1 to t_2 causing current to flow in the circuit leading to joule heating.

S.35.6. The torque is

$$\tau = \mu B \sin\theta = iAB \sin\theta = iAB = ia^2B \quad \ldots(1)$$

where μ is the magnetic moment of the dipole, i is the current, the area $A=a^2$, and $\theta=90°$ is the angle between B and the surface area.

$$i = \frac{Blv}{R} = \frac{Bl\omega r}{R} \quad \ldots(2)$$

where l is the length of the loop and R the resistance.

$$R = \frac{l}{(\sigma)(rt)} \quad \ldots(3)$$

Using (3) in (2),

$$i = B\sigma t\omega r^2 \quad \ldots(4)$$

Using (4) in (1)

$$\tau = B^2a^2\omega r^2\sigma t \quad \ldots(5)$$

36 INDUCTANCE

36.1. The induced emf is given by

$$E=L\frac{di}{dt}$$

Let the current change at the rate of

$$\frac{di}{dt}=\frac{E}{L}=\frac{100 \text{ volt}}{10 \text{ henry}}=10 \text{ amp/sec.}$$

36.2. (*a*) Let the two coils having self-inductances L_1 and L_2 be connected in series, a great distance apart. The equivalent self-inductance of the network is defined as the ratio of the total induced emf between the terminals of the network, to the rate of change of current responsible for the emf.

emf in coil 1 = self-induced emf

$$=L_1\frac{di}{dt}$$

$$\text{emf in coil 2} = L_2\frac{di}{dt}$$

$$\text{Net emf} = (L_1+L_2)\frac{di}{dt}$$

From its definition, the equivalent self-inductance is

$$L=L_1+L_2 \qquad \ldots(1)$$

(*b*) Separation should be large so that mutual-inductance M may not be present, otherwise formula (1) would be modified to

$$L=L_1+L_2\pm 2M \qquad \ldots(2)$$

The signs + or − correspond to the sence of the windings being the same or opposite, respectively.

36.3. If i is the total current,

$$E=-L\frac{di}{dt} \qquad \ldots(1)$$

Let the inductances L_1 and L_2 be in parallel.

Current in one inductance, say L_1 will be $i/2$, that is

$$i_1=i/2$$

$$E_1 = -L_1 \frac{di_1}{dt} = -L_1 \frac{d}{dt}(\tfrac{1}{2}i) = -\frac{L_1}{2}\frac{di}{dt}. \qquad \ldots(2)$$

But $\quad E = E_1$

Comparing (1) and (2), we get

$$L = L_1/2$$

36.4. The magnetic induction between two parallel wires carrying equal currents i in opposite directions at a distance x from one wire is

$$B = \frac{\mu_0 i}{2\pi}\left(\frac{1}{x} + \frac{1}{d-x}\right)$$

The flux is given by

$$\phi_B = \int Bl\,dx = \frac{\mu_0 il}{2\pi}\int_a^{d-a}\left(\frac{1}{x} + \frac{1}{d-x}\right)dx$$

$$= \frac{\mu_0 il}{2\pi}\Big[\ln x - \ln(d-x)\Big]_a^{d-a}$$

$$= \frac{\mu_0 il}{2\pi}\left[\ln\frac{x}{d-x}\right]_a^{d-a} = \frac{\mu_0 il}{2\pi}\left[\ln\left(\frac{d-a}{a}\right) - \ln\frac{a}{(d-a)}\right]$$

$$= \frac{\mu_0 il}{2\pi}\ln\left(\frac{d-a}{a}\right)^2 = \frac{\mu_0 il}{\pi}\ln\frac{d-a}{a}$$

$$\therefore \quad L = \frac{\phi_B}{i} = \frac{\mu_0 l}{\pi}\ln\frac{d-a}{a}.$$

36.5. For the toroid,

$$L = \frac{\mu_0 N^2 h}{2\pi}\ln\frac{b}{a}. \qquad \ldots(1)$$

where N is the total number of turns.

Let $\quad b = a + \Delta$

where Δ is a small quantity,

$$\ln\frac{b}{a} = \ln\frac{a+\Delta}{a} = \ln\left(1 + \frac{\Delta}{a}\right) \simeq \frac{\Delta}{a}$$

$$\therefore \quad L = \frac{\mu_0 N^2 h\Delta}{2\pi a} \qquad \ldots(2)$$

But $h\Delta = A$, ...(3)

where A is the area of cross-section.

$$2\pi a = l \qquad \ldots 4$$

where l is the length

$$N = nl \qquad \ldots 5$$

where n is the number of turns/unit length.

Use (3), (4) and (5) in (2) to find

$$L = \mu_0 n^2 l A$$

an expression appropriate for the solenoid. Thus, if the solenoid is long and thin enough the equation for the inductance of a toroid reduces to that for a solenoid.

36.6. The inductance/unit length, for the solenoid near its center is

$$\frac{L}{l} = \mu_0 n^2 A$$

where n is the number of turns/unit length and A is the area of cross-section.

$$n = \frac{1.0 \text{ meter}}{(0.1 \text{ in})(2.54 \times 10^{-2} \text{ meter/in})} = 394$$

$$A = \pi(0.02 \text{ meter})^2 = 1.256 \times 10^{-3} \text{ meter}^2$$

$$\frac{L}{l} = (4\pi \times 10^{-7} \text{ weber/amp-m})(394)^2(1.256 \times 10^{-3} \text{ meter}^2)$$

$$= 0.245 \times 10^{-3} \text{ h/meter}$$

$$= 0.245 \text{ mh/meter.}$$

36.7. Inductance L is given by, $L = \dfrac{N\phi_B}{i}$.

$$\phi_B = \frac{Li}{N} = \frac{(8 \times 10^{-3} \text{ h})(5 \times 10^{-3} \text{ amp})}{400} = 1.0 \times 10^{-7} \text{ weber}$$

36.8. (*a*) Inductance of the toroidal core is

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a}$$

where N is the number of turns, h is the height of the windings, b is the outer radius, and a is the inner radius.

As the cross-section is square,

$$h = b - a = 12 \text{ cm} - 10 \text{ cm} = 2 \text{ cm.}$$

Diameter of each wire is $d=0.04$ in$=0.1016$ cm.

$$N=\frac{2\pi a}{d}=\frac{(2\pi)(10\text{ cm})}{(0.1016\text{ cm})}=618$$

$$L=\frac{(4\pi\times10^{-7}\text{ weber/amp-m})(618)^2(0.02\text{ meter})}{2\pi}\ln\frac{12}{10}$$

$$=0.28\times10^{-3}\text{ h}$$
$$=0.28\text{ mh}$$

(*b*) Perimeter of each turn for the square cross-section is

$$4h=(4)(2\text{ cm})=8\text{ cm}$$

Therefore, length of the wire

=(number of turns)(perimeter)

=(618)(8 cm)=4944 cm

=(4944 cm)(3.28×10^{-2} ft/cm)=162 ft

Resistance of wire is

$$R=\frac{(1.0\text{ ohm})}{(160\text{ ft})}(162\text{ ft})=1.0\text{ ohm}$$

Time constant, $\tau=\dfrac{L}{R}=\dfrac{0.28\times10^{-3}\text{ h}}{1.0\text{ ohm}}$

$$=2.8\times10^{-4}\text{ sec.}$$

36.9. The current i at time t is related to steady state value i_0 by

$$i=i_0\ (1-e^{-t/\tau})$$

where τ is the inductive constant (we have dropped off the subscript L for the time constant for brevity).

$$e^{-t/\tau}=1-\frac{i}{i_0}=1-\frac{1}{3}=\frac{2}{3}$$

or $\qquad e^{t/\tau}=1.5$

whence $\qquad \tau=\dfrac{t}{\ln 1.5}=\dfrac{5.0\text{ sec}}{\ln 1.5}=12.3\text{ sec.}$

36.10. It is required to have, $i/i_0=1.0-0.001=0.999$

From $i=i_0\ (1-e^{-t/\tau})$

$$i/i_0=0.999=1-e^{-t/\tau}$$

or $\qquad e^{-t/\tau}=0.001$ or $e^{t/\tau}=1000$

$\therefore \qquad t/\tau=\ln 1000=6.9$

36.11. (*a*) For an *L-R* circuit

$$i=(E/R)(1-e^{-t/\tau}) \quad \ldots(1)$$

Joule heating is

$$dP_J=i^2R=(E^2/R)(1-e^{-t/\tau})^2 \quad \ldots(2)$$

where use has been made of (1).

The total energy transformed to joule heat in time $t=\tau$ is

$$P_J=\int dP_J=\frac{E^2}{R}\int_0^\tau (1-e^{-t/\tau})^2\,dt$$

$$=\frac{E^2}{R}\left[\int_0^\tau dt-2\int_0^\tau e^{-t/\tau}\,dt+\int_0^\tau e^{-2t/\tau}dt\right]$$

$$=\frac{E^2}{R}\left[\tau-2\tau(1-e^{-1})+\frac{\tau}{2}(1-e^{-2})\right]$$

$$=\frac{E^2}{R}\tau\left[1-2(1-0.368)+\frac{1}{2}(1-0.135)\right]$$

$$=0.168\,\frac{E^2\tau}{R}.$$

(*b*) The energy stored in the magnetic field at time t is,

$$U_B(t)=\frac{1}{2}Li^2=\frac{1}{2}\frac{E^2}{R^2}(1-e^{-t/\tau})^2 \quad \ldots(3)$$

Set $t=\tau$ and $L/R=\tau$

$$U_B(\tau)=\frac{1}{2}\frac{\tau E^2}{R}(1-2e^{-1}+e^{-2})=0.2\,\frac{\tau E^2}{R}$$

(*c*) Set $t=\infty$ in (3) to find the equilibrium energy stored in the magnetic field.

$$U_B(\infty)=\frac{1}{2}L\frac{E^2}{R^2}=0.5\frac{\tau E^2}{R}$$

36.12. $i=i_0(1-e^{-t/\tau})$

Differentiating with respect to time,

$$\frac{di}{dt}=\frac{i_0}{\tau}e^{-t/\tau}$$

Initial rate of increase is

$$\left.\frac{di}{dt}\right|_{t=0}=\left.\frac{i_0}{\tau}e^{-t/\tau}\right|_{t=0}=\frac{i_0}{\tau}$$

Suppose $di/dt = i_0/\tau$ = constant, at all times, an assumption which is incorrect. Then

$$\int dt = \frac{\tau}{i_0} \int_0^{i_0} di$$

or $$t = \frac{\tau i_0}{i_0} = \tau$$

36.13. $i = i_0(1 - e^{-t/\tau})$

$$\tau = \frac{L}{R} = \frac{50 \times 10^{-3}\ \text{h}}{180\ \text{ohm}} = 2.78 \times 10^{-4}\ \text{sec}$$

$$\frac{t}{\tau} = \frac{0.001\ \text{sec}}{2.78 \times 10^{-4}\ \text{sec}} = 3.6$$

$$i_0 = \frac{E}{R} = \frac{50\ \text{volt}}{180\ \text{ohm}} = 0.278\ \text{amp}$$

$$\frac{di}{dt} = \frac{i_0}{\tau}\ e^{-t/\tau}$$

$$\therefore \quad \left.\frac{di}{dt}\right|_{t=0.001\ \text{sec}} = \frac{0.278\ \text{amp}}{2.78 \times 10^{-4}\ \text{secc}}\ e^{-3\cdot 6} = 27.3\ \text{amp/sec.}$$

36.14. $\tau = \dfrac{L}{R} = \dfrac{2.0\ \text{h}}{10\ \text{ohm}} = 0.2$ sec. ...(1)

$$\frac{t}{\tau} = \frac{0.1\ \text{sec}}{0.2\ \text{sec}} = 0.5 \quad ...(2)$$

(*a*) Rate at which energy is stored in the magnetic field is

$$\frac{dU_B}{dt} = Li\frac{di}{dt} \quad ...(3)$$

But, $$i = i_0(1 - e^{-t/\tau}) \quad ...(4)$$

$$\frac{di}{dt} = \frac{i_0}{\tau}\ e^{-t/\tau} \quad ...(5)$$

Using (4) and (5) in (3),

$$\frac{dU_B}{dt} = L\frac{i_0^2}{\tau}(1 - e^{-t/\tau})\ e^{-t/\tau} \quad ...(6)$$

Also, $$i_0 = \frac{E}{R} = \frac{100\ \text{volt}}{10\ \text{ohm}} = 10\ \text{amp} \quad ...(7)$$

$$\therefore \left.\frac{du_B}{dt}\right|_{t=0.1\text{ sec}} = \frac{(2.0\text{ h})(10\text{ amp})^2}{(0.2\text{ sec})}(1-e^{-0.5})\,e^{-0.5}$$

$$=240\text{ joules/sec}$$

where we have used (6), (7), (1) and (2).

(*b*) Joule heat is produced at the rate

$$P_J = i^2R = i_0^2R(1-e^{-t/\tau})^2$$
$$=(10\text{ amp})^2(10\text{ ohm})(1-e^{-0.5})^2$$
$$=155\text{ joules/sec}$$

(*c*) Rate at which energy is delivered by the battery

$$=(240+155)\text{ joules/sec}$$

$$=395\text{ joules/sec.}$$

36.15. (*a*) The equilibrium current is

$$i_0=\frac{E}{R}=\frac{100\text{ volt}}{10\text{ ohm}}=10\text{ amp.}$$

(*b*) Energy stored in the magnetic field due to current i_0 is

$$U_B=\tfrac{1}{2}Li_0^2=\tfrac{1}{2}(2.0\text{ h})(10\text{ amp})^2=100\text{ joules.}$$

36.16. Joule heat produced in resistor is

$$U_J=\int_0^\infty P_J dt=\int_0^\infty i^2R\,dt. \qquad \ldots(1)$$

When the switch is thrown to *b*, the current decays, being governed by

$$i=i_0e^{-t/\tau} \qquad \ldots(2)$$

Using (2) in (1),

$$U_J=\int_0^\infty i_0^2Re^{-2t/\tau}\,dt=i_0^2R\int_0^\infty e^{-2t/\tau}dt=-\frac{i_0^2}{2}R\tau e^{-2t/\tau}\Big|_0^\infty$$

$$\therefore \qquad U_J=\tfrac{1}{2}i_0^2R\tau \qquad \ldots(3)$$

But $\qquad \tau=L/R \qquad \ldots(4)$

Using (4) in (3)

$U_J=\tfrac{1}{2}Li_0^2=U_B$, the energy stored in the inductor.

36.17. The energy density in magnetic field is given by

$$u_B=\frac{1}{2}\frac{B^2}{\mu_0} \qquad \ldots(1)$$

where B is the magnetic induction. At the center of a loop of radius R the induction is

$$B=\frac{\mu_0 i}{2R} \qquad \ldots(2)$$

Using (2) in (1)

$$u_B=\frac{1}{8}\frac{\mu_0 i^2}{R^2}=\frac{(4\pi\times10^{-7}\text{ weber/amp-meter})(100\text{ amp})^2}{(8)(0.05\text{ meter})^2}$$

$$=0.63\text{ joule/meter}^3$$

36.18. According to Example 9 of Chapter 34, in the hydrogen atom the electron circulates around the nucleus in a path of radius R of 5.1×10^{-11}m at a frequency ν of 6.8×10^{15} rev/sec. The current due to the circulating electron is

$$i=e\nu=(1.6\times10^{-19}\text{ coul})(6.8\times10^{15}\text{ rev/sec})$$

$$=1.1\times10^{-3}\text{ amp.}$$

At the center of the orbit,

$$B=\frac{\mu_0 i}{2R}=\frac{(4\pi\times10^{-7}\text{ weber/amp-m})(1.1\times10^{-3}\text{ amp})}{(2)(5.1\times10^{-11}\text{ meter})}$$

$$=13.5\text{ weber/meter}^2$$

The magnetic energy density at the center of a circulating electron in the hydrogen atom is

$$u_B=\frac{1}{2}\frac{B^2}{\mu_0}=\frac{(13.5\text{ weber/meter}^2)^2}{(2)(4\pi\times10^{-7}\text{ weber/amp-m})}$$

$$=7.3\times10^7\text{ joules/meter}^3.$$

36.19. Energy density stored at any point is

$$u_B=\frac{1}{2}\frac{B^2}{\mu_0} \qquad \ldots(1)$$

Now, B at a distance r from the center of a long cylindrical wire of radius b, where $r<b$, can be calculated by Ampere's law (see Example 1 of Chapter 34).

$$\oint B.dl=\mu_0 i_0$$

$$(B)(2\pi r)=\mu_0 i\,\frac{\pi r^2}{\pi b^2}$$

$$\therefore \qquad B=\frac{\mu_0 ir}{2\pi b^2} \qquad \text{...(2)}$$

Using (2) in (1)

$$u_B=\frac{\mu_0 i^2 r^2}{8\pi^2 b^4}$$

Consider the volume element, $dv=(2\pi rdr)(l)$, where l is the length of wire. Then the total energy in the volume, $v=\pi b^2 l$, of the wire is calculated from

$$U_B=\int u_B dv=\frac{\mu_0 i^2}{8\pi^2 b^4}\int_0^b r^2(2\pi rdr)(l)$$

$$=\frac{\mu_0 i^2 l}{4\pi b^4}\int_0^b r^3 dr=\frac{\mu_0 i^2 l}{16\pi}$$

Therefore, magnetic energy per unit length stored in the wire is

$$\frac{U_B}{l}=\frac{\mu_0 i^2}{16\pi}$$

36.20. By Problem 36.19, magnetic energy per unit length stored in the wire is

$$\frac{U_B}{l}=\frac{\mu_0 i^2}{16\pi}$$

Therefore, magnetic energy in length l of the wire is

$$U_B=\frac{\mu_0 i^2 l}{16\pi} \qquad \text{...(1)}$$

But $\quad U_B=\frac{1}{2}Li^2 \qquad \text{...(2)}$

Comparing (1) and (2)

$$L=\frac{\mu_0 l}{8\pi}$$

an expression which is independent of the wire diameter.

36.21. (*a*) By Problem 36.19, magnetic energy of the wire is

$$\frac{U_B}{l}=\frac{\mu_0 i^2}{16\pi}=\frac{(4\pi\times 10^{-7}\text{ weber/amp-m})(10\text{ amp})^2}{16\pi}$$

$$=2.5\times 10^{-6}\text{ joules/meter.}$$

(*b*) For the co-axial cable, in the space between the two conductors the total magnetic energy stored per unit length (See Example 5 of Chapter 36) is

$$\frac{U_B}{l}=\frac{\mu_0 i^2}{4\pi}\ \ln\ \frac{b}{a}$$

$$U_B=\frac{(4\pi\times10^{-7}\ \text{weber/amp-m})(10\ \text{amp})^2}{4\pi}\ \ln\ \frac{4.0}{1.0}$$

$$=14\times10^{-6}\ \text{joules/mete.}$$

$$\text{(c)}\ u_B=\ \frac{1}{2}\frac{B^2}{\mu_0} \qquad \ldots(1)$$

The magnetic induction within the outer conductor is given by Problem 34.6 (*c*) and is

$$B=\frac{\mu^0 i}{2\pi r}\ \frac{(c^2-r^2)}{(c^2-b^2)} \qquad \ldots(2)$$

Consider the volume element dv within the outer cylinder symmetrical about the axis of the co-axial cable.

$$dv=(2\pi\ rdr)\ l \qquad \ldots(3)$$

where l is the length of the cable.

$$U_B=\int u_B dv=\int\left(\frac{1}{2}\ \frac{B^2}{\mu_0}\right)(2\pi lr)dr \qquad \ldots(4)$$

where we have used (1) and (3).

Using (2) in (4)

$$U_B=\frac{\mu_0 l i^2}{4\pi(c^2-b^2)^2}\int_b^c \frac{1}{r}(c^2-r^2)^2 dr$$

$$\therefore\qquad \frac{U_B}{l}=\frac{\mu_0 i^2}{4\pi(c^2-b^2)^2}\left[c^4\int_b^c \frac{dr}{r}-2c^2\int_b^c rdr+\int_b^c r^3 dr\right]$$

$$=\frac{\mu_0 i^2}{4\pi(c^2-b^2)^2}\left[c^4\ \ln\ \frac{c}{b}-c^2(c^2-b^2)+\frac{1}{4}\ (c^4-b^4)\right]$$

$$=\frac{\mu_0 i^2}{4\pi}\left[\frac{\ln\ c/b}{(1-b^2/c^2)^2}-\frac{3-(b^2/c^2)}{4\ (1-b^2/c^2)}\right]$$

$$=\frac{(4\pi\times10^{-7}\ \text{weber/amp-m})(10\ \text{amp})^2}{4\pi}$$

$$\left(\frac{\ln\ (5/4)}{(1-16/25)^2}-\frac{3-(16/25)}{4(1-16/25)}\right)$$

$$=0.83\times10^{-6}\ \text{joules/meter.}$$

36.22. (*a*) Magnetic energy density,

$$u_B=\frac{\mu_0 i^2}{16\pi}=\frac{(4\pi\times10^{-7}\text{ weber/amp-m})(10\text{ amp})^2}{16\pi}$$

$$=2.5\times10^{-6}\text{ joule/meter}^3$$

(*b*) Electrical energy density,

$$u_E=\tfrac{1}{2}\varepsilon_0 E^2$$

where E is the electric field.

$$E=\frac{V}{l}=\frac{iR}{l}=iR$$

since $l=1.0$ meter.

$$R=1.0\text{ ohm}/1000\text{ ft}=\frac{1.0\text{ ohm}}{(1000\text{ ft})(0.3048\text{ meter/ft})}$$

$$=3.28\times10^{-3}\text{ ohm/meter}$$

$$u_E=\tfrac{1}{2}\varepsilon_0 i^2R^2$$

$$=\tfrac{1}{2}(8.9\times10^{-12}\text{ coul}^2/\text{nt-m}^2)(10\text{ amp})^2(3.28\times10^{-3}\text{ ohm/meter})^2$$

$$=4.8\times10^{-15}\text{ joules/meter}^3.$$

36.23. The magnetic energy density is

$$u_B=\frac{1}{2}\frac{B^2}{\mu_0}$$

The electric energy density is

$$u_E=\tfrac{1}{2}\varepsilon_0 E^2$$

Set $u_E=\mu_B$

$$\therefore\ \tfrac{1}{2}\varepsilon_0 E^2=\tfrac{1}{2}\frac{B^2}{\mu_0}$$

or $$E=\frac{B}{\sqrt{\varepsilon_0\mu_0}}=\frac{0.5\text{ weber/meter}^2}{\sqrt{(8.9\times10^{-12}\text{ coul}^2/\text{nt-m}^2)(4\pi\times10^{-7}\text{ weber/amp-m})}}$$

$$=1.5\times10^8\text{ volt/meter.}$$

SUPPLEMENTARY PROBLEMS

S.36.1. (*a*) By Problem 36.2, the equivalent self-inductance of two inductors L_1 and L_2 in series is given by

$$L_0=L_1+L_2\pm2M \qquad ...(1)$$

where M is the mutual inductance.

The positive sign before the last term in the right side of (1) is applicable for the arrangement in which the flux linking each coil, due to the current in the other is in the same direction as the flux due to the current in the coil, and the negative sign for the arrangement in which the flux linking each coil due to the current in the other is opposite in direction to the coil's own flux.

If the inductors are far apart then $M=0$.

Setting $L_1=L_2=L$ in (1), we find, $L_0=2L$.

(*b*) If two closely wound coils are placed side by side then almost entire flux set by either coil would link with all the turns of the other. By definition,

$$L_1=\frac{N_1\phi_1}{i_1} \qquad \text{...(2)}$$

$$L_2=\frac{N_2\phi_2}{i_2} \qquad \text{...(3)}$$

$$M=\frac{N_1\phi_{12}}{i_2}=\frac{N_2\phi_{21}}{i_1} \qquad \text{...(4)}$$

where N_1 and N_2 are the number of turns of coils 1 and 2 respectively; ϕ_{21} is the flux linking circuit 2 due to current i_2 in circuit 1, and ϕ_{12} is the flux linking circuit 1 due to current i_2 in circuit 2.

Now, $\phi_{12}=\phi_2$ ...(5)

$\phi_{21}=\phi_1$ ...(6)

Hence, $M=\dfrac{N_1\phi_2}{i_2}$ ...(7)

$M=\dfrac{N_2\phi_1}{i_1}$ ...(8)

Multiplying (7) and (8)

$$M^2=\left(\frac{N_1\phi_2}{i_2}\right)\left(\frac{N_2\phi_1}{i_1}\right)=\left(\frac{N_1\phi_1}{i_1}\right)\left(\frac{N_2\phi_2}{i_2}\right)=L_1L_2$$

or $M=\sqrt{L_1L_2}$ ...(9)

where use has been made of (2) and (3).

Setting $L_1=L_2=L$ in (9), we get

$M=L$

Using these values in (1),

$L_0=0$ or $4L$

depending on the direction of winding.

S.36.2. For condition (I), time $t=0$

(a) Applying the loop theorem to the left loop,

$$E-R_1 i_1=0$$

$$i_1=\frac{E}{R_1}=\frac{10 \text{ volt}}{5 \text{ ohm}}=2 \text{ amp}$$

(b) Applying the loop theorem to the outer loop,

$$E-\frac{L di_2}{dt}-R_2 i_2=0$$

But $\left.\frac{di_2}{dt}\right|_{t=0}=\left.\frac{E}{L}e^{-R_2 t/L}\right|_{t=0}=\frac{E}{L}$

$\therefore \; E-L\frac{E}{L}-R_2 i_2=0 \quad$ or $i_2=0$

(c) Applying the junction theorem to the junction of R_1 and R_2,

$$i=i_1+i_2=2 \text{ amp}+0=2 \text{ amp}$$

(d) $V_{(R_2)}=i_2 R_2=0.$

(e) $V_L=L\frac{di_2}{dt}=E=10$ volt

where use has been made of (b).

(f) $L\frac{di_2}{dt}=E-R_2 i_2=10 \text{ volt}-\text{zero}=10$ volt

$\therefore \quad \frac{di^2}{dt}=\frac{10 \text{ volt}}{5 \text{ henry}}=2$ amp/sec.

For condition (II), time $t=\infty$.

(a) $i_1=\frac{E}{R_1}=\frac{10 \text{ volt}}{5 \text{ ohm}}=2$ amp·

(b) $E-L\frac{di_2}{dt}-R_2 i_2=0$

But $\left.\frac{di_2}{dt}\right|_{t=\infty}=\left.\frac{E}{L}e^{-R_2 t/L}\right|_{t=\infty}=0$

$\therefore \quad E-i_2 R_2=0$

or $\qquad i_2=\frac{E}{R_2}=\frac{10 \text{ volt}}{10 \text{ ohm}}=10$ amp.

(c) $i=i_1+i_2=2 \text{ amp}+1 \text{ amp}=3$ amp.

(*d*) $V_2 = i_2R_2 = (1 \text{ amp})(10 \text{ ohm}) = 10 \text{ volt}.$

(*e*) $V_L = L\dfrac{di_3}{dt} = 0$

where use has been made of (*b*).

(*f*) $\dfrac{di_3}{dt} = 0$

where use has been made of (*b*).

S.36.3. (*a*) At $t=0$, the current i_3 through L will be zero. Hence

$$i_1 = i_2 \qquad \ldots(1)$$

Applying loop theorem to the left loop,

$$E - i_1R_1 - i_2R_2 = 0 \qquad \ldots(2)$$

Using (1) in (2),

$$i_1 = i_2 = \frac{E}{R_1 + R_2} = \frac{100 \text{ volt}}{(10+20) \text{ ohm}} = 3.3 \text{ amp}.$$

(*b*) At $t=\infty$, i_3 will have a non-zero value.

From junction theorem,

$$i_2 + i_3 = i_1 \qquad \ldots(3)$$

Using loop theorem for the left loop,

$$E - i_1R_1 - i_2R_2 = 0 \qquad \ldots(4)$$

Using loop theorem for the right loop,

$$-R_3i_3 - L\frac{di_3}{dt} + i_2R_2 = 0 \qquad \ldots(5)$$

But $\left.\dfrac{di_3}{dt}\right|_{t=\infty} = 0$

Therefore, (5) reduces to

$$-R_3i_3 + i_2R_2 = 0$$

or
$$i_3 = \frac{R_2i_2}{R_3} \qquad \ldots(6)$$

Solving (3), (4) and (6),

$$i_1 = \frac{E(R_2+R_3)}{R_1R_2 + R_2R_3 + R_3R_1}$$

$$= \frac{(100 \text{ volt})(20\Omega + 30\Omega)}{(10\Omega)(20\Omega) + (20\Omega)(30\Omega) + (30\Omega)(10\Omega)} = 4.5 \text{ amp}$$

$$i_2 = \frac{ER_3}{R_1R_2 + R_2R_3 + R_3R_1}$$

$$= \frac{(100 \text{ volt})(30\Omega)}{(10\Omega)(20\Omega) + (20\Omega)(30\Omega) + (30\Omega)(10\Omega)} = 2.7 \text{ amp.}$$

(*c*) $i_1 = 0$

The inductor would retain the same current i_3 when the switch is just opened. From (6),

$$i_3 = (2.7 \text{ amp}) \frac{(20\ \Omega)}{(30\ \Omega)} = 1.8 \text{ amp.}$$

Also, from (3) with $i_1 = 0$,

$$i_2 = -i_3 = -1.8 \text{ amp.}$$

(*d*) A long time after the switch is opened the inductor would have lost all the current and $i_3 = 0$. Also $i_1 = 0$ as in (*c*).

S.36.4. The magnetic energy stored in a coaxial cable is given by

$$U_M = \frac{\mu_0 i^2 l}{4\pi} \ln \frac{b}{a} \qquad \ldots(1)$$

where i is the current and l is the length of the cable (see Example 36.5 of Chapter 36).

Also, its capacitance is given by

$$C = \frac{2\pi\varepsilon_0 l}{\ln (b/a)} \qquad \ldots(2)$$

By Problem, the electric and magnetic energies are equal.

$$U_E = \frac{1}{2} E^2 C = \frac{\mu_0 i^2 l}{4\pi} \ln \frac{b}{a} \qquad \ldots(3)$$

where use has been made of (1).

Using (2) in (3)

$$\frac{1}{2} E^2 \frac{2\pi\varepsilon_0 l}{\ln(b/a)} = \frac{\mu_0 i^2 l}{4\pi} \ln \frac{b}{a}$$

or

$$\frac{E}{i} = R = \frac{1}{2\pi} \sqrt{\frac{\mu_0}{\varepsilon_0}} \ln \frac{b}{a}$$

$$= \frac{1}{2\pi} \sqrt{\frac{4\pi \times 10^{-7} \text{ weber/amp-m}}{8.85 \times 10^{-12} \text{ coul}^2/\text{nt-m}^2}} \ln \frac{b}{a}$$

$$= \frac{377}{2\pi} \ln \frac{b}{a} \text{ ohms.}$$

S.36.5. The dimensions and units of some of the electrical quantities are tabulated below for ready reference.

Quantity	*Dimensions*	*Unit*
Coulomb	Q	coulomb
Current	$T^{-1}Q$	ampere
Voltage	$ML^2T^{-2}Q^{-1}$	volt
Resistance	$ML^2T^{-1}Q^{-2}$	ohm
Inductance	ML^2Q^{-2}	henry
Capacitance	$M^{-1}L^{-2}T^2Q^2$	farad
Magnetic flux	$ML^2T^{-1}Q^{-1}$	weber

(*a*) [coulomb-ohm-meter/weber]

$$=\frac{[Q][ML^2T^{-1}Q^{-2}][L]}{[ML^2T^{-1}Q^{-1}]}=[L]=[\text{meter}]$$

(*b*) [volt-second]$=[ML^2T^{-2}Q^{-1}][T]=[ML^2T^{-1}Q^{-1}]=$[weber]

(*c*) [coulomb-ampere/farad] $=\dfrac{[Q][T^{-1}Q]}{[M^{-1}L^{-2}T^2Q^2]}=[ML^2T^{-3}]=$[watt]

(*d*) $\dfrac{[\text{kilogram-volt-meter}^2]}{[\text{henry-ampere}]}=\dfrac{[M][ML^2T^{-2}Q^{-1}][L^2]}{[ML^2Q^{-2}]^2(T^{-1}Q)^2}$

$$=(Q)=(\text{coulomb})$$

(*e*) (henry/farad)$^{1/2}=\dfrac{(ML^2Q^{-2})^{1/2}}{(M^{-1}L^{-2}T^2Q^2]^{1/2}}=(ML^2T^{-1}Q^{-2})=$(ohm)

37 MAGNETIC PROPERTIES OF MATTER

37.1. (*a*) Magnetic moment due to the current i through a loop is

$$\mu = iA \qquad \ldots(1)$$

where A is the area of the loop.

Setting, $A = \pi r^2$, were r is the radius of earth, (1) becomes

$$\mu = \pi r^2 i$$

or $$i = \frac{\mu}{\pi r^2} = \frac{6.4 \times 10^{21} \text{ amp-m}^2}{\pi(6.4 \times 10^6 \text{ meter})^2} = 5 \times 10^7 \text{amp.}$$

(*b*) Yes.

(*c*) No.

37.2. Volume element in spherical polar coordinates is given by

$$dv = 2\pi r^2 dr \sin\theta \, d\theta$$

If the electron carries charge e and its volume is $(4/3)\ \pi R^3$, R being its radius, then the charge associated with the volume element is

$$dq = \frac{3e dv}{4\pi R^3} = \frac{3e}{4\pi R^3}(2\pi r^2 \, dr \sin\theta \, d\theta)$$

or $$dq = \frac{3er^2 \, dr \sin\theta \, d\theta}{2R^3} \qquad \ldots(1)$$

The charge element dq goes around in a loop of radius $r \sin\theta$ about the axis of rotation, the area of the loop being

$$dA = \pi(r \sin\theta)^2 \qquad \ldots(2)$$

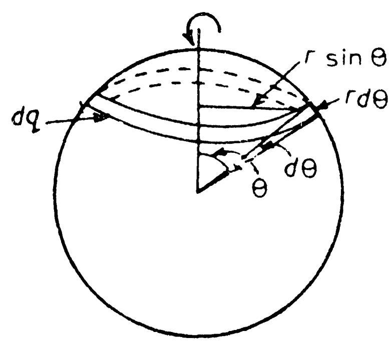


Fig 37.2.

If the rotational frequency of the electron is ν then any charge element dq would also be circulating about the axis of rotation with the same frequency $\nu = \frac{\omega}{2\pi}$. The contribution to the magnetic moment due to an infinitesimal current loop is

$$d\mu = \nu\ dq\ dA$$

$$= \frac{\omega}{2\pi} \quad \frac{3er^2\ dr \sin\theta\ d\theta\ \pi r^2 \sin^2\theta}{2R^3}$$

$$= \frac{3e}{4R^3}\ \omega r^4\ dr \sin^3\theta\ d\theta$$

The magnetic momont of the electron is found out by integrating the above expression

$$\mu = \int d\mu = \frac{3e\omega}{4R^3} \int_0^R r^4 dr \int_0^\pi \sin^3\theta\ d\theta$$

$$= \frac{3e\omega}{4R^3}\ \frac{R^5}{5}\ \frac{4}{3} = \frac{e\omega R^2}{5}$$

$$\therefore \qquad \mu = \frac{e\omega R^2}{5} \qquad \ldots(3)$$

The mechanical angular momentum (spin) based on the classical model is given by

$$L = I\omega$$

where $I=(2/5)\ mR^2$, is the rotational inertia of a sphere of mass m, rotating about an axis nassing through its center. We can then write

$$L = \frac{2}{5} m\omega R^2 \qquad \ldots(4)$$

Dividing (3) by (4),

$$\frac{\mu}{L} = \frac{e}{2m}$$

or
$$\frac{e}{m} = \frac{2\mu}{L} \qquad \ldots(5)$$

Expression (5) is in disagreement with experiment, the observed value being $e/m=\mu/L$.

37.3. (*a*) Electric field strength at a distance r is

$$E = \frac{q}{4\pi\ \epsilon_0 r^2}$$

$$= \frac{(1\text{-}6\times 10^{-19}\text{ coul})(9\times 10^9\text{ nt-m}^2/\text{coul}^2)}{(1.0\times 10^{-10}\text{ meter})^2}$$

$$= 1.44\times 10^{11}\text{ volt/meter},$$

(*b*) Magnetic induction at a distance r is

$$B=\frac{\mu_0\mu}{2\pi r^3}$$

$$=\frac{(4\pi\times10^{-7}\text{ weber/amp-m})(1.4\times10^{-26}\text{ amp-m}^2)}{(2\pi)(1.0\times10^{-10}\text{ meter})^3}$$

$$=2.8\times10^{-3}\text{ weber/meter}^2.$$

37.4. (*a*) In the Rowland ring, a toroidal coil is wound around the iron specimen in the form of a ring and originally unmagnetized With the iron core absent, a current i set up in the coil causes a field of induction within the toroid given by

$$B_0=\mu_0 ni=\frac{\mu_0 Ni}{2\pi r} \qquad ...(1)$$

where n is the number of turns per unit length of the toroid. N is the total number of turns and r is the mean radius of the ring.

From (1) we have

$$i=\frac{2\pi rB_0}{\mu_0 N}=\frac{(2\pi)(5.5\times10^{-2}\text{ meter})(2\times10^{-4}\text{ weber/m}^2)}{(4\pi\times10^{-7}\text{ weber/amp-m})(400)}$$

$$=0.14\text{ amp.}$$

(*b*) Because of the iron core the actual value of the induction will be B, given by

$$B=B_0+B_M \qquad ...(2)$$

By Problem, $B_M=800\ B_0$. ...(3)

Using (3) in (2),

$$B=801B_0=(801)(2\times10^{-4}\text{ weber/meter}^2)$$

$$=0.1602\text{ weber/meter}^2 \qquad ...(4)$$

The charge q flowing through the secondary coil is given by

$$q=\frac{BN'A}{R} \qquad ...(5)$$

Where N' is the number of turns of the secondary coil, R the resistance of the secondary coil and A the cross-section area of the toroid.

$$A=\pi d^2/4 \qquad ...(6)$$

With the diameter of the cross-sectional area being

$$d=6\text{ cm}-5\text{ cm}=1.0\text{ cm}=0.01\text{ meter}$$

$$A=\frac{\pi}{4}(0.01\text{ meter})^2=7.85\times10^{-5}\text{ meter}^2. \qquad ...(7)$$

Using (4), (7) and the values $N'=50$ and $R=8.0$ ohms, in (5), we find

$$q=\frac{(0.16\ \text{weber/meter}^2)(50)(7.85\times10^{-5}\ \text{meter}^2)}{(8.0\ \text{ohm})}$$

$$=7.85\times10^{-5}\ \text{coul.}$$

37.5. (*a*) The dipole moment of the bar is

$$\mu_b=N\mu \qquad \ldots(1)$$

Here N, the number of atoms of iron in volume $v=(5\ \text{cm})(1\text{cm}^2)=5\ \text{cm}^3$, is given by

$$N=\frac{N_0\rho v}{M}$$

where $N_0=6\times10^{23}$ is the Avagadro's number, ρ the density of iron and M the atomic weight of iron. We have

$$N=\frac{(6\times10^{23}/\text{mole})(7.9\ \text{gm/cm}^3)(5\ \text{cm}^3)}{(55.85)}$$

$$=4.2\times10^{23}.$$

$$\mu_b=(4.2\times10^{23})(1.8\times10^{-23}\ \text{amp-m}^2)=7.6\ \text{amp-m}^2$$

(*b*) The magnitude of the torque is given by

$$\tau=\mu B\sin\theta=(7.6\ \text{amp-m}^2)(1.5\ \text{weber})(\sin\ 90°)$$

$$=11.4\ \text{nt-meter.}$$

37.6. The atoms of a diamagnetic material do not have permanent magnetic moments, the individual magnetic moments of orbital electrons neutralizing each other. However, in the presence of an external magnetic field, a magnetic moment is induced in the atom. The phenomenon of diamagnetism can be explained in terms of electromagnetic induction.

Consider a typical electron revolving in the xy plane as in Textbook Fig. 37.7 (*a*) and (*b*). Let an external magnetic field be switched on along the z-direction. In the absence of an external magnetic field the centripetal force which enables the electron of mass m and charge e to move around in a circular orbit of radius r is given by r

$$F_0=\frac{mv_0^2}{r} \qquad \ldots(1)$$

Let the field B be increasing at a rate dB/dt. This will induce an electric field E around the path. In accordance with Faraday's law of induction, the induced emf is

$$E=-\frac{d\phi_B}{dt}$$

The flux through the loop is

$$\phi_B = B(\pi r^2)$$

$$E = \oint \mathbf{E}.d\mathbf{l} = -\pi r^2 \frac{dB}{dt}$$

$$(E)(2\pi r) = -\pi r^2 \frac{dB}{dt}$$

or $$E = -\frac{1}{2} r \frac{dB}{dt} \qquad \ldots(2)$$

The additional force acting on the electron is

$$m\frac{dv}{dt} = qE = \frac{1}{2} er \frac{dB}{dt} \qquad \ldots(3)$$

where (2) has been used and the sign has been ignored. From (3) we find a connection between the change in v and the change in B,

$$dv = \frac{er}{2m} dB \qquad \ldots(4)$$

Suppose v increases. As the radius of the orbit remains constant, the increase in centripetal force is provided by the magnetic field.

(The speed of electron v will increase or decrease depending on the direction of the magnetic induction B).

Call Δv the net change in v during the process the field attains the final value B.

$$\Delta v = \int_{v_0}^{v_0+\Delta v} dv = \frac{er}{2m} \int_0^B dB = \frac{erB}{2m} \qquad \ldots(5)$$

The magnetic moment arising from the circulating electron is

$$\mu = iA = (ev_0)(\pi r^2) = \frac{e\omega_0}{2\pi} \pi r^2$$

or $$\mu = \tfrac{1}{2} er^2 \omega_0$$

Change in magnetic moment, holding r as constant, is given by

$$\Delta\mu = \tfrac{1}{2} er^2 \Delta\omega \qquad \ldots(6)$$

But from (5) we have

$$\Delta\omega = \frac{\Delta v}{r} = \pm \frac{eB}{2m} \qquad \ldots(7)$$

Using (7) in (6)

$$\Delta\mu = \pm \frac{e^2 r^2 B}{4m}$$

Thus, the effect of applying a magnetic field is to increase or decrease the angular velocity of electron depending on the sense of circulation. This, in turn, causes an increase or decrease of magnetic moment. The change in the magnetic moment is in opposition to the applied field.

37.7. By Problem 37.6

$$\Delta\omega = \pm \frac{eB}{2m}$$

$$\therefore \left| \frac{\Delta\omega}{\omega_0} \right| = \frac{eB}{2m\omega_0}$$

Set $e = 1.6 \times 10^{-19}$ coul ; $m = 9.1 \times 10^{-31}$ kg,

$\omega_0 = 4.3 \times 10^6$ rad/sec ; $B = 2.0$ weber/meter^2.

We have chosen a typical value for B, and the value for ω is corresponding to Bohr's model for hydrogen atom.

$$\left| \frac{\Delta\omega}{\omega_0} \right| = \frac{(1.6 \times 10^{-19} \text{ coul})(2 \text{ weber/meter}^2)}{(2)(9.2 \times 10^{-31} \text{ kg})(4.3 \times 10^{16} \text{ rad/sec})} = 4 \times 10^{-6}$$

Thus, $\Delta\omega \ll \omega_0$

37.8. The spin angular momentum is a vector which points in the direction of the axis of rotation (Fig. 37.8). The direction of μ and magnetic moment and the sense of flow of positive charge in the loop are related by the right-hand-screw rule.

The spin magnetic moment is given by

$$\mu = \frac{1}{2} er^2\omega = \frac{e}{2m}(mvr)$$

Thus, the magnetic moment is proportional to the angular momentum. Designating spin angular momentum by the vector S, we may rewrite the above relation

$$\mu = \frac{e}{2m} \mathbf{S}$$

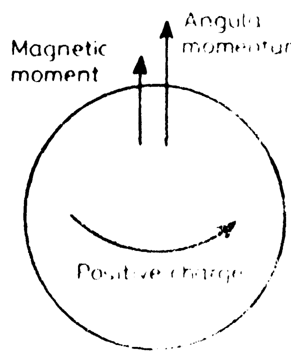


Fig. 37.8

The positive sign in the right hand side shows that the current of the positive charge is in the same direction as the spin motion. The spin and the magnetic moment point in the same direction.

37.9. $B = N \frac{\mu_0 \mu}{2\pi x_3}$

where N is the number of protons in the sample and μ is the magnetic moment of the proton.

Number of water molecules in 1.0 gm of sample

$$=\frac{N_0}{M}=\frac{(6\times 10^{23}/\text{mole})}{(18\text{ gm})}=3.3\times 10^{22}$$

Now, in a molecule of H_2O there aro 10 protons. (2 from hydrogen atoms and 8 from oxygen atom). Therefore, the number of protons in the sample is

$$N=(10)(3.3\times 10^{22})=3.3\times 10^{23}$$

$$B=\frac{(3.3\times 10^{23})(4\pi\times 10^{-7}\text{ weber/amp-m})(1.41\times 10^{-26}\text{ joule/Tesla})}{(2\pi)(5\times 10^{-2}\text{ meter})^3}$$

$$=7.5\times 10^{-6}\text{ weber/meter}^2.$$

37.10. $\nu_c=\dfrac{eB}{2\pi m}$ ...(1)

or $\dfrac{e}{m}=\dfrac{2\pi\nu_c}{B}$ ...(2)

$\nu_P=\dfrac{\mu_s B}{2\pi L_s}$ ...(3)

or $\dfrac{2\pi}{B}=\dfrac{\mu_s}{\nu_p L_s}$ ...(4)

Using (4) in (2),

$$\frac{e}{m}=\frac{\nu_c\mu_s}{\nu_p L_s}$$

37.11. (*a*) As the field in the median plane of a dipole is only half as large as on the axis,

$B=\dfrac{1}{2}\,\dfrac{\mu_0\mu}{2\pi a^3}$ ...(1)

$$B=\frac{(4\pi\times 10^{-7}\text{ weber/amp-m})(1.8\times 10^{-23}\text{ amp-m}^2)}{(4\pi)(1.0\times 10^{-10}\text{ meter})^3}$$

$$=1.8\text{ weber/meter}^2.$$

(*b*) Energy required to turn a second similar dipole end for end in this field is

$$U_B=2\mu B$$
$$=(2)(1.8\times 10^{-23}\text{ amp-m}^2)(1.8\text{ weber/meter}^2)$$
$$=6.5\times 10^{-23}\text{ joules.}$$

At room temperature, the mean kinetic energy of translation is

$$U_T=\frac{3}{2}kT=\left(\frac{3}{2}\right)(1.38\times 10^{-23}\text{ joule/°K})(300\text{°K})$$
$$=6\times 10^{-21}\text{ joule.}$$

Thus, $U_T \simeq 100\, U_B$. Consequently, the energy exchanges in collisions can destroy the alignment of the dipoles with the external field.

37.12. In Fig. 37.12 the line ab lies in the interface between the two media 1 and 2. Let B_1 and B_2 make angles θ_1 and θ_2 respectively,

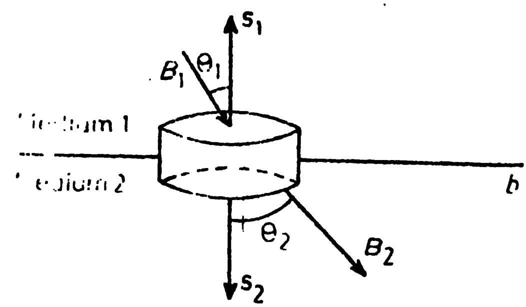


Fig. 37.12

with the normal to the interface. We shall now apply Gauss' theorem, to the pill box shown in the diagram

$$\int_S B\cdot ds = 0$$

or $$0 = \int B\cdot ds = -B_1 s_1 + B_2 s_2$$

or $$0 = (-B_1 \cos\theta_1 + B_2 \cos\theta_2)\, s$$

whence. $$B_1 \cos\theta_1 = B_2 \cos\theta_2$$

showing thereby that the normal component of the induction has the same value on each side of the surface.

37.13. Consider the rectangular path as shown in Fig. 37.13. Since the closed path encloses zero current, it follows that the line integral of H around the path is zero.

$$\oint \mathbf{H}.dl = 0$$

In the evaluation of the above integral the contribution due to the paths de and cf can be ignored. Thus,

$$0 = \int_c^i H_1.dl + \int_e^f H_2.dl$$

or $$0 = (-H_1 \sin\theta_1 + H_2 \sin\theta_2)\, l$$

or $$H_1 \sin\theta_1 = H_2 \sin\theta_2$$

Thus the tangential component of H has the same value on each side of the surface.

By Problem $F_E = NF_B$ ...(2)

Now, $F_B = Bev = Be\omega r$ ...(3)

$F_C = m\omega^2 r$ ...(4)

Combining (1), (2), (3) and (4)

$$Be\omega r\,(N \pm 1) = m\omega^2 r$$

or $$\omega = \frac{Be}{m}(N \pm 1)$$

(*b*) $$\omega_1 = \frac{Be}{m}(N+1)$$

$$= \frac{(0.427 \text{ weber/meter}^2)(1.6 \times 10^{-19} \text{ coul})(100+1)}{(9.1 \times 10^{-31} \text{ kg})}$$

$$= 758 \times 10^{10} \text{ rad/sec.}$$

$$\omega_2 = \frac{Be}{m}(N-1)$$

$$= \frac{(0.427 \text{ weber/meter}^2)(1.6 \times 10^{-19} \text{ coul})(100-1)}{(9.1 \times 10^{-31} \text{ kg})}$$

$$= 743 \times 10^{10} \text{ rad/sec.}$$

S.37.4. The magnet will tend to align its axis antiparallel to the direction of *B* but is opposed by the torque due to its own weight. The torque due to magnetic field is

$$\tau_B = \mu B \cos\theta \quad ...(1)$$

Here θ is the angle between the axis of the magnet and vertical and μ is the magnetic moment, given by

$$\mu = 2ml \quad ...(2)$$

where $2l$ is the distance between the poles of the magnet and m is the pole strength. In Fig. S.37.4 the torque τ_B acts in the counterclockwise sense.

The torque τ_g due to the weight acting at the center of the magnet is given by

$$\tau_g = Mgl \sin\theta \quad ...(3)$$

acting in the clockwise sense.

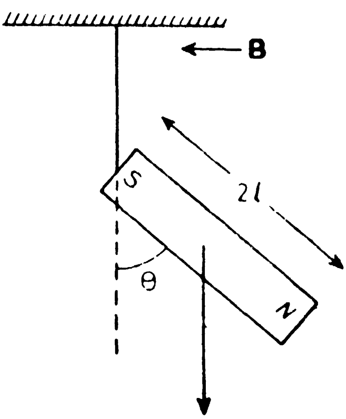


Fig S.37.4

For rotational equilibrium we have the condition,

$$\tau_g = \tau_B \quad ...(4)$$

$$Mgl \sin\theta = \mu B \cos\theta$$

or $$\tan\theta = \frac{\mu B}{Mgl} \quad ...(5)$$

Thus the magnet is oriented at an angle θ with the vertical given by (5) with the north pole moving away from the direction of B, whilst the string remains in the vertical direction.

S.37.5. The reluctance S is given by

$$S=\frac{1}{\mu_0 A}\left[\frac{l_1}{\mu_r}+l_2\right]$$

where μ_r is the relative permeability, l_1 is the flux path, l_2 is the gap length and A is the area of cross-section.

$$S=\frac{1}{(4\pi\times10^{-7}\text{ weber/amp-m})\,A}\left[\frac{1.0\text{ meter}}{5000}+0.01\text{ meter}\right]$$

$$=\frac{8121}{A}\text{ amp/weber} \qquad \ldots(1)$$

The flux,

$$\phi=BA=(1.8\text{ weber/m}^2)\,A=1.8A\text{ weber} \qquad \ldots(2)$$

Magnetomotive force $(mmf)=Ni$ ampere-turn ...(3)

$$\phi=(mmf/S)=Ni/S$$

or $$i=\frac{\phi S}{N}=\frac{(1.8A\text{ weber})}{500}\left(\frac{8121}{A}\text{ amp/weber}\right)=29.2\text{ amp.}$$

38 ELECTROMAGNETIC OSCILLATIONS

38.1. $C_1=5\ \mu f$; $C_2=2\ \mu f$; $L=10$ mh

(*a*) LC_1 combination:

Resonant frequency, $\nu_1=\dfrac{1}{2\pi\sqrt{LC_1}}$

$$=\frac{1}{2\pi\sqrt{(10\times10^{-3}\ \text{h})(5\times10^{-6}\ \text{f})}}$$

$=714$ cycles/sec.

(*b*) LC_2 combination:

Resonant frequency, $\nu_2=\dfrac{1}{2\pi\sqrt{LC_2}}$

$$=\frac{1}{2\pi\sqrt{(10\times10^{-3}\ \text{h})(2\times10^{-6}\ \text{f})}}$$

$=1126$ cycles/sec

(*c*) LC_1C_2 combination with C_1 and C_2 in parallel:

Equivalent capacitance of C_1 and C_2 is

$$C=C_1+C_2=5\ \mu f+2\ \mu f=7\ \mu f$$

Resonent frequency, $\nu_3=\dfrac{1}{2\pi\sqrt{LC}}$

$$=\frac{1}{2\pi\sqrt{(10\times10^{-3}\ \text{h})(7\times10^{-6}\ \text{f})}}$$

$=602$ cycles/sec

(*d*) LC_1C_2 combination C_1 and C_2 in series:

Equivalent capacitance of C_1 and C_2 is

$$C=\frac{C_1C_2}{C_1+C_2}=\frac{(5\ \mu f)(2\ \mu f)}{(5\ \mu f+2\ \mu f)}=1.43\ \mu f$$

Resonent frequency, $\nu_4=\dfrac{1}{2\pi\sqrt{LC}}$

$$=\frac{1}{2\pi\sqrt{(10\times10^{-3}\ \text{h})(1.43\times10^{-6}\ \text{f})}}$$

$=1333$ cycles/sec.

38.2. Frequency of oscillation is $\nu=\dfrac{1}{2\pi\sqrt{LC}}$.

$$\therefore \qquad C=\frac{1}{4\pi^2\nu^2L}$$

$$=\frac{1}{(4\pi^2)(1.0\times10^6\ \text{sec}^{-1})^2(1.0\times10^{-3}\ \text{h})}$$

$$=2.5\times10^{-11}\ \text{farad.}$$

Thus, by combining a capacitor of $C=2.5\times10^{-11}$ farad with the given inductance $L=1.0$ mh, we can get oscillations at 1.0×10^6 cycles/sec.

38.3. Frequency range extends from $\nu_1=2\times10^5$ cycles/sec to $\nu_2=4\times10^5$ cycles/sec.

The frequency range is,

$$\Delta\nu=\nu_2-\nu_1$$

$$=4\times10^5\ \text{cycles/sec}-2\times10^5\ \text{cycles/sec}$$

$$=2\times10^5\ \text{cycles/sec.}$$

The angular range on the knob is $\Delta\theta=180°$. Therefore, the rotation through 1° corresponds to a frequency change of 2×10^5 cycles/(sec-degree) rotation. For any rotation of $\theta°$, the frequency would change by $2\times10^5\ \theta°/180°$ cycles/sec. Thus, the angle θ and ν would be related by

$$\nu=\nu_1+\frac{2\times10^5\ \theta°}{180°}=2\times10^5\left(1+\frac{\theta}{180°}\right)\text{cycles/sec} \quad ...(1)$$

$$C=\frac{1}{4\pi^2\nu^2L}$$

$$=\frac{1}{(4\pi^2)(2\times10^5\ \text{cycles/sec})^2(1.0\times10^{-3}\ \text{h})\left(1+\dfrac{\theta}{180}\right)^2}$$

$$\text{or} \qquad C=\frac{634\times10^{-12}}{\left(1+\dfrac{\theta}{180}\right)^2}\ \text{farad}=\frac{634}{\left(1+\dfrac{\theta}{180}\right)^2}\ \mu\mu f$$

Figure 38.3 shows the plot of C as a fraction of angle for 180° rotation.

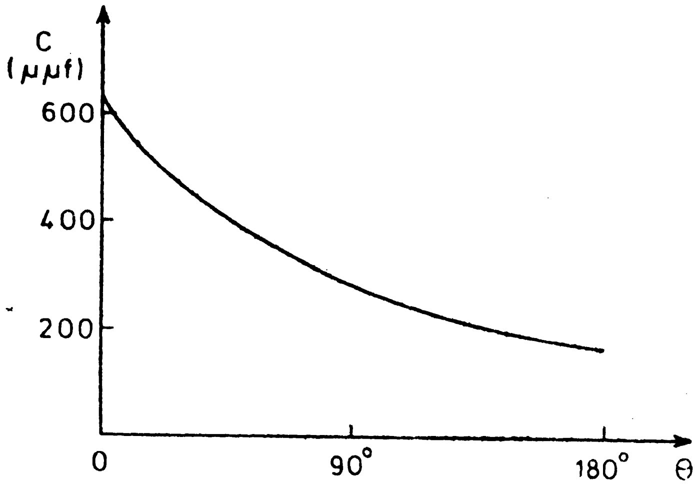


Fig. 38.3

38.4. For an *LCR* circuit resonant frequency is given by

$$v=\frac{\omega'}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{1}{LC}-\left(\frac{R}{2L}\right)^2} \quad \ldots(1)$$

Squaring and re-arranging (1)

$$\frac{1}{LC}=4\pi^2\nu^2+\frac{R^2}{4L^2} \quad \ldots(2)$$

For the given data for ν, R and L, the second term on the right hand side of (2) is quite small compared to the first term. Eq. (2) then reduces to

$$\frac{1}{LC}=4^2\pi^2\nu^2$$

or $$C=\frac{1}{4\pi^2\nu^2L}=\frac{1}{(4\pi^2)(60 \text{ cycles/sec})^2(10 \text{ h})}$$
$$=7\times10^{-7} \text{ farad}=0.07\ \mu\text{f}.$$

The given *LCR* circuit is as good as resistanceless *LC* circuit.

38.5. The potential drop across the capacitor is

$$V_C=q/C$$

and the potential drop across the inductance is

$$V_L=L\frac{di}{dt}$$

Applying the loop theorem to the *LC* circuit,

$$V_C + V_L = 0$$

i.e. $$\frac{q}{C} + L\frac{di}{dt} = 0$$

But $$\frac{di}{dt} = \frac{d}{dt}\left(\frac{dq}{dt}\right) = \frac{d^2q}{dt^2}$$

$$\therefore \quad \frac{q}{C} + L\frac{d^2q}{dt^2} = 0$$

This is the Textbook Eq. 38.5.

38.6. Conservation of energy demands that the applied emf, $E = E_m \cos \omega'' t$ be equal to the sum of the potential drop across *L*, *C* and *R* of the circuit.

$$\therefore \quad L\frac{di}{dt} + \frac{q}{C} + iR = E_m \cos \omega'' t \qquad \ldots(1)$$

But $$i = \frac{dq}{dt} \qquad \ldots(2)$$

$$\therefore \quad \frac{di}{dt} = \frac{d}{dt}\left(\frac{dq}{dt}\right) = \frac{d^2q}{dt^2} \qquad \ldots(3)$$

Using (2) and (3) in (1)

$$L\frac{d^2q}{dt^2} + \frac{q}{C} + R\frac{dq}{dt} = E_m \cos \omega'' t$$

which is in agreement with Textbook Eq. 38.12 of the textbook.

38.7. The stored energy in the capacitor is

$$U_E = \frac{q^2}{2C} \qquad \ldots(1)$$

and the maximum stored energy in the capacitor is

$$U_{E,max} = \frac{q_m^2}{2C} \qquad \ldots(2)$$

By Problem, $U_E = \frac{1}{2} U_{E,max}$ $\qquad \ldots(3)$

$$\therefore \quad \frac{q^2}{2C} = \frac{1}{2}\,\frac{q_m^2}{2C} \qquad \ldots(4)$$

where use has been made of (1) and (2)

$$\therefore \quad q = \frac{q_m}{\sqrt{2}} \qquad \ldots(5)$$

But the oscillation amplitude of charge is

$$q = q_m e^{-Rt/2L}$$

whence $e^{Rt/L}=\frac{q_m^2}{q^2}=2$

where use has been made of (5).

$$\therefore \quad \frac{Rt}{L}=\ln 2$$

or $$t=\frac{L}{R}\ln 2=0.69\tau_L$$

where we have used the fact that the induction time-constant $\tau_L=L/R$ and $\ln 2=0.693$.

38.8. The undamped resonent frequency is given by

$$\nu=\frac{1}{2\pi\sqrt{LC}} \quad \text{.. (1)}$$

The damped frequency is given by

$$\nu'=\frac{1}{2\pi}\sqrt{\frac{1}{LC}-\left(\frac{R}{2L}\right)^2}$$

By Problem, $\frac{\nu-\nu^1}{\nu}=\frac{0.01}{100}=1\times10^{-4}$.

But $\frac{\nu-\nu'}{\nu}=1-\frac{\nu'}{\nu}$

$$=1-\sqrt{1-\frac{R^2C}{4L}}=1\times10^{-4}.$$

Solving for R, we get,

$$R=\sqrt{\frac{8L}{C}\times10^{-4}}$$

$$=\sqrt{(8)(10\times10^{-3}\text{ h})(10^{-4})/(10^{-6}\text{ f})}$$

$$=\sqrt{8}\text{ ohm}=2.8\text{ ohm}.$$

38.9. $$\omega=\frac{1}{\sqrt{LC}} \quad \text{...(1)}$$

$$\omega'=\sqrt{\frac{1}{LC}-\left(\frac{R}{2L}\right)^2} \quad \text{...(2)}$$

$$\frac{\omega-\omega'}{\omega}=1-\frac{\omega'}{\omega}=1-\sqrt{1-\frac{R^2C}{4L}}\simeq1-\left(1-\frac{R^2C}{8L}\right)$$

$$=\frac{R^2C}{8L} \quad \text{..(3)}$$

where use has been made of (1) and (2) and we have expended the radical binomially.

$$q=q_m e^{-Rt/2L}$$

whence, $$t=\frac{2L}{R}\ln\left(\frac{q_m}{q}\right)=\frac{2L}{R}\ln 2 \qquad \ldots(4)$$

$$t=\frac{n}{\nu}=2\pi n\sqrt{LC}=\frac{2L}{R}\ln 2$$

or $$\frac{CR^2}{L}=\frac{(\ln 2)^2}{(\pi n)^2} \qquad \ldots(5)$$

Use (5) in (3) to find,

$$\frac{\omega-\omega'}{\omega}=\frac{(\ln 2)^2}{8\pi^2 n^2}=\frac{(0.69315)^2}{(8)(9.8696)\,n^2}=\frac{0.006085}{n^2}$$

$$\simeq\frac{0.0061}{n^2}.$$

38.10. $q = q_m e^{-Rt/2L}\cos\omega' t$

Differencitate with respect of time to obtain

$$i=\frac{dq}{dt}=q_m\left(-\frac{R}{2L}\,e^{-Rt/2L}\cos\omega' t-\omega'\sin\omega' t e^{-Rt/2L}\right)$$

$$=-q_m\omega^1 e^{-Rt/2L}\left(\frac{R}{2L\omega'}\cos\omega' t+\sin\omega' t\right)$$

$$=-q_m\omega' e^{-Rt/2L}(\tan\phi\cos\omega' t+\sin\omega' t)$$

where we have set $\frac{R}{2L\omega'}$,$=\tan\phi$. This gives

$$i=\frac{-q_m\omega' e^{-Rt/2L}}{\cos\emptyset}(\sin\phi\cos\omega' t+\cos\phi\sin\omega' t)$$

$$=-\frac{q_m\omega' e^{-Rt/2L}}{\cos\phi}\sin(\omega' t+\phi)$$

But for low damping, the resistance R is small.

Hence, $\frac{R}{2L\omega'}$, $\to 0$ and $\phi\to 0$

$\therefore \quad \cos\phi\to 1$

So, $\quad i=-q_m\omega' e^{-Rt/2L}\sin(\omega^1 t+\phi)$

38.11. If $U_{E,\,max}$ is the initial maximum energy and U_E the energy at any time then change in energy

$$\Delta U = U_{E,max} - U_E = \frac{q^2_m}{2C} - \frac{q^2}{2C} \qquad ...(1)$$

But, amplitude of charge oscillation is

$$q = q_m\, e^{-Rt/2L}$$

$$\therefore \qquad q^2 = q^2_m\, e^{-Rt/L} \qquad ...(2)$$

Using (2) in (1)

$$\Delta U = \frac{q^2_m}{2C} - \frac{q^2_m\, e^{-Rt/L}}{2C} = \frac{q^2_m}{2C}\left(1 - e^{-Rt/L}\right)$$

$$= U_{E,\,max}(1 - e^{-Rt/L})$$

Set $U_{E,\,max} = U$, then

$$\frac{\Delta U}{U} = 1 - e^{-Rt/L}$$

One cycle implies that $t = \dfrac{1}{\nu} = \dfrac{2\pi}{\omega}$

$$\therefore \quad \frac{\Delta U}{U} = 1 - e^{-2\pi R/\omega L} = 1 - \left(1 - \frac{2\pi R}{\omega L} + \ldots\right)$$

$$= \frac{2\pi R}{\omega L}$$

38.12. $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E_m \cos \omega'' t \qquad ...(1)$$

Let $\qquad q = a \sin(\omega'' t - \phi) \qquad ...(2)$

Then, $\qquad \dfrac{dq}{dt} = a\omega'' \cos(\omega'' t - \phi) \qquad ...(3)$

and $\qquad \dfrac{d^2q}{dt^2} = -a\omega''^2 \sin(\omega'' t - \phi). \qquad ...(4)$

Substituting (2), (3) and (4) in (1),

$$-a\omega''^2 L \sin(\omega'' t - \phi) + a\omega'' R \cos(\omega'' t - \phi) + \frac{a}{C} \sin(\omega'' t - \phi)$$

$$= E_m \cos \omega'' t \qquad ...(5)$$

Expanding the sine and cosine functions

$$-a\omega''^2 L(\sin\omega''t\cos\phi-\cos\omega''t\sin\phi)+a\omega''R(\cos\omega''t\cos\phi$$
$$+\sin\omega''t\sin\phi)+\frac{a}{C}(\sin\omega''t\cos\phi-\cos\omega''t\sin\phi)$$
$$=E_m\cos\omega''t \qquad \ldots(6)$$

Comparing the coefficients of $\sin\omega''t$,

$$-a\omega''^2 L\cos\phi+a\omega''R\sin\phi+\frac{a}{C}\cos\phi=0$$

Re-arranging the terms and cancelling the common factor a,

$$\tan\phi=\frac{1}{\omega''R}\left(\omega''^2L-\frac{1}{C}\right) \qquad \ldots(7)$$

Comparing the coefficients of $\cos\omega''t$ in (6)

$$a\omega''^2L\sin\phi+a\omega''R\cos\phi-\frac{a}{C}\sin\phi=E_m$$

or
$$a=\frac{E_m}{\left(\omega''^2L-\frac{1}{C}\right)\sin\phi+\omega''R\cos\phi} \qquad \ldots(8)$$

From (7) using trignometric identities we easily find

$$\sin\phi=\frac{\omega''^2L-1/C}{\sqrt{(\omega''R)^2+(\omega''^2L-1/C)^2}} \qquad \ldots(9)$$

$$\cos\phi=\frac{\omega''R}{\sqrt{(\omega''R)^2+(\omega''^2L-1/C)^2}} \qquad \ldots(10)$$

Substituting (9) and (10) in (8) and simplifying,

$$a=q_m=\frac{E_m}{\sqrt{(\omega''^2L-1/C)^2+(\omega''R)^2}}$$

Maximum value of q_m is conditioned by setting $\dfrac{dq_m}{d\omega''}=0$

$$\frac{dq_m}{d\omega''}=-\frac{1}{2}E_m\frac{[4\omega''L(\omega''^2L-1/C)+2R^2\omega'']}{[(\omega''^2L-1/C)^2+(\omega''R)^2]}=0$$

Cancelling the denominator as well the common factor $-E_m$,

$$\omega''(2L^2\omega''^2-2L/C+R^2)=0$$

The nontrivial solution which gives q_m a maximum is

$$\omega''=\sqrt{\frac{1}{LC}-\frac{R^2}{2L^2}}$$

38.11. If $U_{E,\,max}$ is the initial maximum energy and U_E the energy at any time then change in energy

$$\Delta U = U_{E,max} - U_E = \frac{q^2{}_m}{2C} - \frac{q^2}{2C} \qquad ...(1)$$

But, amplitude of charge oscillation is

$$q = q_m\, e^{-Rt/2L}$$

$$\therefore \qquad q^2 = q^2{}_m\, e^{-Rt/L} \qquad ...(2)$$

Using (2) in (1)

$$\Delta U = \frac{q^2{}_m}{2C} - \frac{q^2{}_m\, e^{-Rt/L}}{2C} = \frac{q^2{}_m}{2C}\left(1 - e^{-Rt/L}\right)$$

$$= U_{E,\,max}(1 - e^{-Rt/L})$$

Set $U_{E,\,max} = U$, then

$$\frac{\Delta U}{U} = 1 - e^{-Rt/L}$$

One cycle implies that $t = \frac{1}{\nu} = \frac{2\pi}{\omega}$

$$\therefore \qquad \frac{\Delta U}{U} = 1 - e^{-2\pi R/\omega L} = 1 - \left(1 - \frac{2\pi R}{\omega L} + \ldots\right)$$

$$= \frac{2\pi R}{\omega L}$$

38.12. $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E_m \cos \omega'' t \qquad ...(1)$$

Let $\qquad q = a \sin(\omega'' t - \phi) \qquad ...(2)$

Then, $\quad \frac{dq}{dt} = a\omega'' \cos(\omega'' t - \phi) \qquad ...(3)$

and $\quad \frac{d^2q}{dt^2} = -a\omega''^2 \sin(\omega'' t - \phi). \qquad ...(4)$

Substituting (2), (3) and (4) in (1),

$$-a\omega''^2 L \sin(\omega'' t - \phi) + a\omega'' R \cos(\omega'' t - \phi) + \frac{a}{C}\sin(\omega'' t - \phi)$$

$$= E_m \cos \omega'' t \qquad ...(5)$$

Expanding the sine and cosine functions

$$-a\omega''^2L\,(\sin\omega''t\cos\phi-\cos\omega''t\sin\phi)+a\omega''R(\cos\omega''t\cos\phi$$
$$+\sin\omega''t\sin\phi)+\frac{a}{C}(\sin\omega''t\cos\phi-\cos\omega''t\sin\phi)$$
$$=E_m\cos\omega''t \qquad \ldots(6)$$

Comparing the coefficients of $\sin\omega''t$,

$$-a\omega''^2L\cos\phi+a\omega''R\sin\phi+\frac{a}{C}\cos\phi=0$$

Re-arranging the terms and cancelling the common factor a,

$$\tan\phi=\frac{1}{\omega''R}\left(\omega''^2L-\frac{1}{C}\right) \qquad \ldots(7)$$

Comparing the coefficients of $\cos\omega''t$ in (6)

$$a\omega''^2L\sin\phi+a\omega''R\cos\phi-\frac{a}{C}\sin\phi=E_m$$

or
$$a=\frac{E_m}{\left(\omega''^2L-\frac{1}{C}\right)\sin\phi+\omega''R\cos\phi} \qquad \ldots(8)$$

From (7) using trignometric identities we easily find

$$\sin\phi=\frac{\omega''^2L-1/C}{\sqrt{(\omega''R)^2+(\omega''^2L-1/C)^2}} \qquad \ldots(9)$$

$$\cos\phi=\frac{\omega''R}{\sqrt{(\omega''R)^2+(\omega''^2L-1/C)^2}} \qquad \ldots(10)$$

Substituting (9) and (10) in (8) and simplifying,

$$a=q_m=\frac{E_m}{\sqrt{(\omega''^2L-1/C)^2+(\omega''R)^2}}$$

Maximum value of q_m is conditioned by setting $\frac{dq_m}{d\omega''}=0$

$$\frac{dq_m}{d\omega''}=-\frac{1}{2}E_m\frac{[4\omega''L(\omega''^2L-1/C)+2R^2\omega'']}{[(\omega''^2L-1/C)^2+(\omega''R)^2]}=0$$

Cancelling the denominator as well the common factor $-E_m$,

$$\omega''\,(2L^2\omega''^2-2L/C+R^2)=0$$

The nontrivial solution which gives q_m a maximum is

$$\omega''=\sqrt{\frac{1}{LC}-\frac{R^2}{2L^2}}$$

38.13. The *LCR* circuit will oscillate with maximum response i.e. the maximum amplitude of the current oscillations occur when the frequency ω'' of the applied emf is exactly equal to the natural (undamped) frequency ω of the system.

$$\omega''=\omega=\frac{1}{\sqrt{LC}}$$

$$\therefore \qquad \nu''=\frac{\omega''}{2\pi}=\frac{1}{2\pi}\frac{1}{\sqrt{(1.0\text{ h})(20\times10^{-6}\text{ f})}}=35.5\text{ cycles/sec.}$$

The amplitude i_m of the current oscillations is given by

$$i_m=\frac{E_m}{\sqrt{(\omega''L-1/\omega''C)+R^2}}$$

At resonance, $\omega''=\omega$

and $$i_m=\frac{E_m}{R}$$

By Problem,

$$i_m=\frac{E_m}{\sqrt{(\omega''L-1/\omega''C)^2+R^2}}=\frac{1}{2}\frac{E_m}{R}$$

Squaring and simplifying

$$\left(\omega''L-\frac{1}{\omega''C}\right)^2=3R^2$$

or $$\omega''L-\frac{1}{\omega''C}=\pm\sqrt{3}\,R$$

or $$\omega''^2LC+\sqrt{3}\,\omega''\,CR-1=0$$

$$\omega''=\pm\frac{\sqrt{3}\,R}{2L}+\sqrt{\frac{3R^2}{4L^2}+\frac{1}{LC}}$$

These are the only possible solutions.

$$\omega''=\pm\frac{\sqrt{3}\,(20\text{ ohm})}{(2)(1.0\text{ h})}+\sqrt{\frac{3}{4}\left(\frac{20\text{ ohm}}{1.0\text{ h}}\right)^2+\frac{1}{(1.0\text{ h})(20\times10^{-6}\text{ f})}}$$

$$\omega_1''=241.6\text{ rad/sec}$$

$$\omega_2''=207\text{ rad/sec.}$$

$$\nu_1''=\frac{\omega_1''}{2\pi}=\frac{241.6\text{ rad/sec}}{2\pi}=38.5\text{ cycles/sec.}$$

$$\nu_2''=\frac{\omega_2''}{2\pi}=\frac{207\text{ rad/sec}}{2\pi}=33.0\text{ cycles/sec.}$$

38.14. The amplitude i_m of the current oscillations is given by

$$i_m=\frac{E_m}{\sqrt{(\omega''L-1/\omega''C)^2+R^2}}$$

At resonance, $\omega''=\omega$ and

$$i_m=\frac{E_m}{R}$$

Set $$i_m=\frac{E_m}{\sqrt{(\omega''L-1/\omega''C)^2+R^2}}=\frac{1}{2}\frac{E_m}{R}$$

Squaring and simplifying

$$\left(\omega''L-\frac{1}{\omega''C}\right)^2=3R^2$$

or $$\omega^{2''}LC\pm\sqrt{3}\,\omega''\,CR-1=0$$

The only acceptable solutions are

$$\omega_1''=\frac{\sqrt{3}}{2}\frac{R}{L}+\sqrt{\frac{3R^2}{4L^2}+\frac{1}{LC}}$$

$$\omega_2''=-\frac{\sqrt{3}}{2}\frac{R}{L}+\sqrt{\frac{3R^2}{4L^2}+\frac{1}{LC}}$$

Subtracting the last equation from the previous one, and dropping off the primes,

$$\Delta\omega=\omega_1-\omega_2=\sqrt{3}\frac{R}{L}$$

$$\therefore\quad \frac{\Delta\omega}{\omega}=\frac{\sqrt{3}R}{\omega L}$$

38.15. Resonant frequency for $L_1C_1R_1$ in series is

$$\omega_1=\frac{1}{\sqrt{L_1C_1}} \qquad \ldots(1)$$

Resonant frequency for $L_2C_2R_2$ in series is

$$\omega_2=\frac{1}{\sqrt{L_2C_2}} \qquad \ldots(2)$$

By Problem, $\omega_1=\omega_2$ ...(3)

that is, $L_1C_1=L_2C_2$ ...(4)

For the two combinations in series, the equivalent inductance is

$$L=L_1+L_2 \qquad \ldots(5)$$

Assuming that the inductances are far apart, the equivalent capacitance is

$$C=\frac{C_1C_2}{C_1+C_2}\cdot \qquad \ldots(6)$$

The resonant frequency is then given by

$$\omega=\frac{1}{\sqrt{LC}}=\frac{1}{\sqrt{(L_1+L_2)C_1C_2/(C_1+C_2)}} \qquad \ldots(7)$$

where use has been made of (5) and (6).

Using (4) in (7), we find upon simplification,

$$\omega=\frac{1}{\sqrt{L_2C_1}}=\frac{1}{\sqrt{L_2C_2}}$$

38.16. Example 5 is concerned with a parallel-plate capacitor with circular plates.

By definition, the displacement current is given by

$$i_d=\varepsilon_0\frac{d\phi_E}{dt}=\varepsilon_0\frac{d}{dt}[(E)(\pi r^2)]$$

$$=\varepsilon_0\pi r^2\frac{dE}{dt}$$

Displacement current density is

$$j_d=\frac{i_d}{\pi r^2}=\varepsilon_0\frac{dE}{dt}$$

38.17. By definition, the displacement current is given by

$$i_d=\varepsilon_0\frac{d\phi_E}{dt}=\varepsilon_0\frac{d}{dt}(EA)=\varepsilon_0A\frac{d\varepsilon}{dt}$$

$$=\varepsilon_0A\frac{d}{dt}\left(\frac{V}{d}\right)=\frac{\varepsilon_0A}{d}\frac{dV}{dt}=C\frac{dV}{dt}$$

where we have used the following formulas for the parallel plate condenser.

$$E=\frac{V}{d}$$

d being the distance of separation of plates, and for the capacitance,

$$C=\frac{\varepsilon_0A}{d}$$

A being the area of cross-section of plates.

38.18. Using the results of Problem 38.17, the displacement current is given bv

$$i_d = C\frac{dV}{dt}$$

$$\therefore \quad \frac{dV}{dt} = \frac{i_d}{C} = \frac{1.0 \text{ amp}}{1.0 \times 10^{-6} \text{ f}} = 10^6 \text{ volts/sec.}$$

Thus, the displacement current of 1.0 amp can be established by changing the potential difference at the rate of 10^6 volt/sec.

38.19. Consider the parallel-plate capacitor with circular plates. By definition the displacement current is given by

$$i_d = \varepsilon_0 \frac{d\phi_E}{dt}$$

In the region $r < R$

$$\phi_E = (E)(\pi r^2)$$

$$\therefore \quad i_d = \varepsilon_0 \frac{d}{dt}(\pi r^2 E) = \varepsilon_0 \pi r^2 \frac{dE}{dt}$$

In the region, $r > R$, $\phi_E = (e)(\pi R^2)$

$$i_d = \varepsilon_0 \frac{d}{dt}(\pi R^2 E) = \varepsilon_0 \pi R^2 \frac{dE}{dt}$$

38.20. (*a*) Ampere's law in its modified form is

$$\oint_l \mathbf{B}.d\mathbf{l} = \mu_0 \int_S (\mathbf{J} + \mathbf{J}_d) .d\mathbf{S} = \mu_0 (i_d + i) \qquad \ldots(1)$$

In the wire conduction current i alone flows whilst within the capacitor C displacement current i_d alone exists. Since the continuity condition says that the total current, conduction current plus displacement current, is constant in a closed circuit, it is sufficient to show that $i_d = i$.

In the gap of the capacitor,

$$E = q/\varepsilon_0 A$$

Differentiating, $dE/dt = \dfrac{1}{\varepsilon_0 A} \, dq/dt = \dfrac{1}{\varepsilon_0 A} \, i$

But $\quad i_d = \varepsilon_0 \dfrac{d\phi_E}{dt} \, E = \varepsilon_0 \dfrac{d\,(EA)}{dt} = \varepsilon_0 A \dfrac{dE}{dt}.$

Combining the last two equations, $i_d = i$.

(*b*) The conduction current (Fig 38.20) flows up the walls of the cavity and the displacement down through the volume of the cavity.

Application of (1) to to region outside the cavity, such as the path r_2 shows that $B=0$. Consequently, the net current enclosed is zero, the conduction current being exactly equal and opposite to the displacement current. Thus, the continuity condition of current is satisfied.

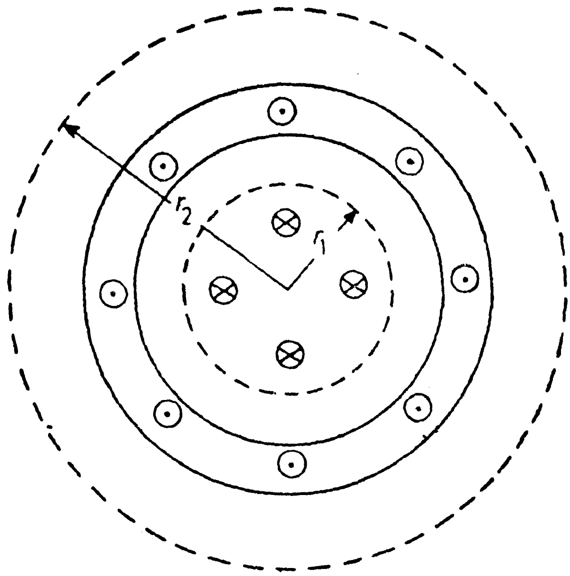


Fig. 38.20

38.21. (*a*) $E=E_m \sin \omega t$

$$\frac{dE}{dt}=E_m\,\omega \cos \omega t$$

$$\left(\frac{dE}{dt}\right)_m=E_m\,\omega$$

Set $\omega=\omega_1$

where the angular frequency for the fundamental mode as in Textbook Fig. 38.8 is given by

$$\omega_1=\frac{1.19c}{a}$$

where c is the speed of light in free space and a is the cavity radius.

$$\omega_1=\frac{(1.19)(3\times 10^8 \text{ meter/sec})}{2.5\times 10^{-2} \text{ meter}}=1.4\times 10^{10} \text{ radians/sec}$$

$$\left(\frac{dE}{dt}\right)_m=(10^4 \text{ volt/meter})(1.4\times 10^{10} \text{ radians/sec}).$$

$$=1.4\times 10^{14} \text{ volt/meter-sec,}$$

(*b*) $B(r)=\frac{1}{2}\mu_0\varepsilon_0 r\frac{dE}{dt}$

By Problem, $\frac{dE}{dt}=\frac{1}{2}\left(\frac{dE}{dt}\right)_m=\frac{1}{2}\times 1.4\times 10^{14}$ volt/meter-sec

$=7\times 10^{13}$ volt/meter-sec.

Set $r=a=2.5\times 10^{-2}$ meter

Also $\mu_0\varepsilon_0=\frac{1}{c^2}$

$$B(r)=\frac{a}{2c^2}\frac{dE}{dt}=\frac{(2.5\times 10^{-2}\text{ meter})(7\times 10^{13}\text{ volt/meter-sec})}{(2)(3\times 10^8\text{ meter/sec})^2}$$

$=10^{-5}$ weber/meter^2

38.22.

	$r<R$	$r>R$
(*a*) $B(r)$	$\frac{\mu_0 ir}{2\pi R^2}$	$\frac{\mu_0 i}{2\pi r}$
(*b*) $E(r)$	$\frac{qr}{4\pi\varepsilon_0 R^3}$	$\frac{\lambda}{2\pi\varepsilon_0 r}$
(*c*) $B(r)$	$\frac{1}{2}\mu_0\varepsilon_0 r\frac{dE}{dt}$	$\frac{\mu_0\varepsilon_0 R^2}{2r}\frac{dE}{dt}$
(*d*) $E(r)$	$-\frac{1}{2}r\frac{dB}{dt}$	$-\frac{1}{2}\frac{R^2 dB}{r dt}$

Maxwell's equations in differential form are

(*i*) $\nabla\times\mathbf{H}=J+\frac{\partial D}{\partial t}$

(*ii*) $\nabla\times E=-\frac{\partial B}{\partial t}$

(*iii*) $\nabla.\mathbf{B}=0$

(*iv*) $\nabla.\mathbf{D}=\rho$

with $B=\mu_0 H,\ D=\varepsilon_0 E$

We shall verify that Maxwell's equations satisfy $B(r)$ and $E(r)$ as given in the above table.

(*a*) $\nabla\times\mathbf{B}=0$; $(r>R)$ ...(1)

$\nabla\times\mathbf{B}=\frac{\mathbf{k}\,i}{\pi R^2}=J$; $(r<R)$ ...(2)

where **k** is a unit vector in the direction of the conductor. Using the cylindrical coordinates, and using only the radial dependent term

$$\nabla \times \mathbf{B} = \mathbf{k}\left(\frac{\partial B}{\partial r} + \frac{B}{r}\right)$$

For $r < R$,

$$B = \frac{\mu_0 i r}{2\pi R^2} = \frac{\mu_0 r J}{2}$$

$$\frac{\partial B}{\partial r} + \frac{B}{r} = \frac{\mu_0 J}{2} + \frac{\mu_0 J}{2} = \mu_0 J$$

$$\therefore \quad \nabla \times \mathbf{B} = \mathbf{k}\mu_0 J = \mu_0 J$$

For $r > R$,

$$B = \frac{i}{2\pi r}$$

$$\therefore \frac{\partial B}{\partial r} + \frac{B}{r} = -\frac{i}{2\pi r^2} + \frac{i}{2\pi r^2} = 0$$

$$\therefore \quad \nabla \times \mathbf{B} = 0$$

(*d*) Assuming electric field is negative in the θ direction and is proportional to r,

$$E_0 = -Ar \qquad \ldots(3)$$

The correctness of this assumption is proved by showing that (3) is satisfied by the magnetic field. Here A is to be determined and is independent of r, θ and z.

$$\nabla \times \mathbf{E} + \mathbf{k}\left(\frac{\partial E}{\partial r} = \frac{E}{r}\right)$$

$$= \mathbf{k}(-A - A) = -\mathbf{k}\ 2A$$

From Maxwell's equation (*ii*)

$$\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t} = -\mathbf{k}\ 2A$$

or $$\frac{\partial B}{\partial t} = 2A$$

or $$A = \frac{1}{2}\frac{\partial B}{\partial t} \qquad \ldots(4)$$

Using (4) in (3),

$$F_\theta = -\frac{r}{2}\frac{\partial B}{\partial t}\ ;\ (r < R) \qquad \ldots(5)$$

The induced electric field assumes a maximum value at a radius R. The electric field does not drop abruptly but dies away at $r > R$ in such a way that the electric field has no curl. To determine the electric field in the space beyond the magnetic field stretches out it is necessary to satisfy the following conditions:

(*i*) E has no curl, (*ii*) E is continuous with the field given by (5) at $r=R$, and (*iii*) $E \to 0$ as $r \to \infty$.

Conditions (1) and (3) are satisfied by

$$E_\theta = -\frac{C}{r} \qquad \ldots(6)$$

where C is a constant. At radius R, (5) and (6) must give the same value of E.

$$\therefore \qquad \frac{C}{R} = \frac{R}{2}\frac{\partial B}{\partial t}$$

or
$$C = \frac{1}{2}R^2 \frac{\partial B}{\partial t} \qquad \ldots(7)$$

Using (7) in (6)

$$E_\theta = -\frac{R^2}{2r}\frac{\partial B}{\partial t} \qquad \ldots(8)$$

SUPPLEMENTARY PROBLEMS

S.38.1. Initial energy stored in the 900 μf capacitor is

$$U_1 = \tfrac{1}{2} C_1 V_1^2 \qquad \ldots(1)$$

If this energy is eventually transferred to the 100 μf capacitor, then

$$U_2 = \tfrac{1}{2} C_2 V_2^2 = \tfrac{1}{2} C_1 V_1^2$$

or
$$V_2 = V_1 \sqrt{\frac{C_1}{C_2}} = (100 \text{ volt}) \sqrt{\frac{900\ \mu\text{f}}{100\ \mu\text{f}}} = 300 \text{ volt.}$$

Thus, the 100 μf capacitor can be charged to 300 volt, but a direct transfer of energy is not possible since the charge from C_1 to C_2 will cease to flow when the capacitors have equal potentials. One can circumvent this difficulty by first converting the electrical energy of C_1 into the magnetic energy of the inductor L by closing the switch S_2 for a specified time and then immediately transfer the magnetic energy to C_2 in the form of electrical energy by opening switch S_2 and closing the switch S_1. It is known that in an LC circuit the transfer of electrical energy into magnetic energy and *vice*

versa takes place in a time $T/4$ where $T=2\pi\sqrt{LC}$ is the time period of the electrical oscillations. The time period for the LC_1 circuit is calculated as $T_1=2\pi\sqrt{LC_1}$

$$=(2\pi)\sqrt{(10\ \text{h})(900\times10^{-6}\ \text{f})}=0.6\ \text{sec.}$$

Similarly, $T_2=2\pi\sqrt{LC_2}=2\pi\sqrt{(10\ \text{h})(100\times10^{-6}\ \text{f})}=0.2$ sec.

The procedure then consists of the following sequence of operations:

Step (*i*) Close S_2 and wait for time $T_1/4$, or 0.15 sec, during which time the 900 μf capacitor is completely discharged and the current in the inductor is fully established producing a magnetic field in the surrounding space.

Step (*ii*) Immediately close S_1 and open S_2 so that now the current in the inductor begins to decrease and after a time $T_2/4=0.05$ sec the magnetic field of L would have disappeared and the 100 μf capacitor fully charged.

Step (*iii*) Immediately after 0.05 sec open the switch S_1. The 100 μf capacitor is now charged to 300 volt.

(*b*) The mechanical analogue consists of the oscillations of a mass m attached to two uncoupled springs of force constants k_1 and k_2 as in Fig. S.38.1, the free ends of the spring being fixed to rigid walls and mass m is free to slide on a frictionless horizontal table. A mechanism R_2 analogous to switch S_2 in the electrical case,

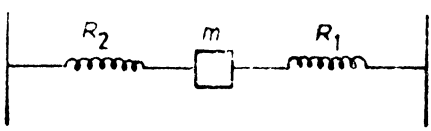


Fig. S.38.1

allows the spring k_1 to be released when it is stretched through a given distance from the equilibrium position. The same mechanism R_2 may be maneouvered to get the mass detached from the spring k_1. Another mechanism R_1 analogous to switch S_1 allows the mass m to be attached to k_2. Furthermore, R_1 may also be used to get the spring k_2 looked up when extended to the maximum.

The procedure for the transfer of deformation energy of spring k_1 (analogous to the electrical energy of C_1) to the spring k_2 (analogous to the magnetic energy of the inductor L) is as follows:

Step (*i*) Operate R_2 so that the mass m is attached to k_1 and moved through a distance A. Then releasc it. Wait for a time $T_1/4$, where $T_1=2\pi\sqrt{m/k_1}$. The mass m would have required maximum kinetic energy.

Step (*ii*) Immediately use mechanism R_2 so that the mass gets detached from k_1, and use R_1 so that m is attached to k_2. Wait for a time $T_2/4$ where $T_2=2\pi\sqrt{m/k_2}$. In this line m would have momentarily come to stop when the spring k_2 is stretched to the maximum.

Step (*iii*) Immediately after time $T_2/4$ use R_1 to get the spring k_2 locked up.

In the table below are compared quantities related to the mechanical and electrical oscillations for the *Mass+Spring* system and the *LC Circuit.*

	Mass+Spring System	*LC Circuit*
(*a*) Nature of motion	Simple harmonic	Simple harmonic
(*b*) Equation of motion	$\frac{1}{2}mv^2+\frac{1}{2}kx^2=\frac{1}{2}kA^2$	$\frac{1}{2}Li^2+\frac{1}{2}\frac{q^2}{C}=\frac{1}{2}\frac{Q^2}{C}$
(*c*) Frequency of oscillation	$\frac{1}{2\pi}\sqrt{\frac{K}{m}}$	$\frac{1}{2\pi}\sqrt{\frac{I}{LC}}$
(*d*) Description of oscillation	$x=A\sin\omega t$, $v=\frac{dx}{dt}$	$q=Q\sin\omega t$, $i=\frac{dq}{dt}$

S.38.2. (*a*) Let the left parallelepiped contain charge q_1 and the right one charge q_2. Applying Gauss law to the left parallelepiped,

$$\varepsilon_0 \oint_1 \mathbf{E}.d\mathbf{S}=q_1 \quad ...(1)$$

Simlarly, for the right parallelepiped,

$$\varepsilon_0 \oint_2 \mathbf{E}.d\mathbf{S}=q_2 \quad ...(2)$$

$$\text{Now, } \varepsilon_0 \oint_1 \mathbf{E}.d\mathbf{S}=\varepsilon_0 \int_{1(NC)} \mathbf{E}.d\mathbf{S}+\varepsilon_0 \int_{1(C)} \mathbf{E}.d\mathbf{S} \quad ...(3)$$

where 1 (*NC*) refers to all the surfaces of the left parallelepiped which are not common with the right side parallelepiped, and 1 (*C*) means common surface. Similarly,

$$\varepsilon_0 \oint_2 \mathbf{E}.d\mathbf{S}=\varepsilon_0 \int_{2(NC)} \mathbf{E}.d\mathbf{S}+\varepsilon_0 \int_{2(C)} \mathbf{E}.d\mathbf{S} \quad ...(4)$$

Adding (1) and (2) and using (3) in (4),

$$q_1+q_2=\varepsilon_0 \int_{1(NC)} \mathbf{E}.d\mathbf{S}+\varepsilon_0 \int_{2(NC)} \mathbf{E}.d\mathbf{S}+\varepsilon_0 \int_{1(C)} \mathbf{E}.d\mathbf{S}+\varepsilon_0 \int_{2(C)} \mathbf{E}.d\mathbf{S} \quad ...(5)$$

The last two terms on the right hand side get cancelled together since the normal to the common surface in the fourth term is opposite to that in the third term and consequently

$$\varepsilon_0 \int_{2(C)} \mathbf{E}.d\mathbf{S} = -\varepsilon_0 \int_{1(C)} \mathbf{E}.d\mathbf{S}$$

$$\therefore\ q_1+q_2=q=\varepsilon_0 \int_{1(NC)} \mathbf{E}.d\mathbf{S} + \varepsilon_0 \int_{2(NC)} \mathbf{E}.d\mathbf{S} = \varepsilon_0 \int_{(Composite)} \mathbf{E}.d\mathbf{S}$$

This proves the self-consistency property of Maxwell's equations.

(*b*) Proceeding along similar lines we can prove the desired property by using $\oint \mathbf{B}.d\mathbf{S}=0$.

S.38.3. (*a*) $\displaystyle\oint_{abcda} \mathbf{E}.d\mathbf{l} = \int_a^b \mathbf{E}.\,d\mathbf{l} + \int_b^c \mathbf{E}.\,d\mathbf{l} + \int_c^d \mathbf{E}.\,d\mathbf{l} + \int_d^a \mathbf{E}.\,d\mathbf{l}$

$$= -\left(\frac{d\phi_B}{dt}\right)_{abcd} \qquad \ldots(1)$$

$$\oint_{befcl} \mathbf{E}.d\mathbf{l} = \int_b^e \mathbf{E}.\,d\mathbf{l} + \int_e^f \mathbf{E}.\,d\mathbf{l} + \int_f^c \mathbf{E}.\,d\mathbf{l} + \int_c^b \mathbf{E}.\,d\mathbf{l}$$

$$= -\left(\frac{d\phi_B}{dt}\right)_{befc} \qquad \ldots(2)$$

Adding (1) and (2),

$$\oint_{abcda} \mathbf{E}.\,d\mathbf{l} + \oint_{befcl} \mathbf{E}.\,d\mathbf{l} = \int_a^b \mathbf{E}.d\mathbf{l} + \int_b^e \mathbf{E}.\,d\mathbf{l} + \int_e^f \mathbf{E}.\,d\mathbf{l} + \int_f^c \mathbf{E}.\,d\mathbf{l}$$

$$+ \int_c^d \mathbf{E}.\,d\mathbf{l} + \int_d^a \mathbf{E}.\,d\mathbf{l} + \int_b^c \mathbf{E}.\,d\mathbf{l} + \int_c^b \mathbf{E}.d\mathbf{l}$$

$$= -\left(\frac{d\phi_B}{dt}\right)_{abcd} - \left(\frac{d\phi_B}{dt}\right)_{befc} \qquad \ldots(3)$$

Now $\displaystyle\int_b^c \mathbf{E}.d\mathbf{l} = -\int_c^b \mathbf{E}.d\mathbf{l}$; $\qquad \ldots(4)$

Further, $\displaystyle -\left(\frac{d\phi_B}{dt}\right)_{abcd} - \left(\frac{d\phi_B}{dt}\right)_{befc}$

$$= -[A_{(abcd)} + A_{(befc)}]\ \frac{dB}{dt}$$

$$= -A_{(aefd)} \frac{dB}{dt}$$

$$= -\left(\frac{d\phi_B}{dt}\right)_{aefd} \qquad \ldots(5)$$

Using (4) and (5) in (3),

$$\int_a^b \mathbf{E}.d\mathbf{l} + \int_b^e \mathbf{E}.d\mathbf{l} + \int_e^f \mathbf{E}.d\mathbf{l} + \int_f^c \mathbf{E}.d\mathbf{l} + \int_c^d \mathbf{E}.d\mathbf{l} + \int_d^a \mathbf{E}.d\mathbf{l}$$

$$= \oint_{aefd} \mathbf{E}.\,d\mathbf{l} = -\left(\frac{d\phi_B}{dt}\right)_{aefd}$$

Thus, the self-consistency of Maxwell's equations is proved.

(*b*) By a procedure similar to the above, we can show by using $\frac{1}{\mu_0} \oint \mathbf{B}.d\mathbf{l} = i + \varepsilon_0 \frac{d\phi_B}{dt}$, the desired property.

S.38.4. According to Gauss' law for electricity,

$$\varepsilon_0 \oint \mathbf{E}.\,d\mathbf{S} = q = \int \rho\, dV \qquad \ldots(1)$$

According to Gauss law for magnetism

$$\oint \mathbf{B}.d\mathbf{S} = \qquad \ldots(2)$$

According to Ampere's law modified by Maxwell

$$\oint \mathbf{B}.\,d\mathbf{l} = \mu_0\varepsilon_0 \frac{d\phi_E}{dt} + \mu_0 i$$

$$= \mu_0\varepsilon_0 \frac{d}{dt} \oint \mathbf{E}.d\mathbf{S} + \mu_0 \int \mathbf{J}.\,d\mathbf{S} \qquad \ldots(3)$$

According to Faraday's law of induction,

$$\oint \mathbf{E}.d\mathbf{l} = -\frac{d\phi_B}{dt} = -\frac{d}{dt} \int \mathbf{B}.d\mathbf{S} \qquad \ldots(4)$$

Equations (1) through (4) are the desired Maxwell's equations.

39 ELECTROMAGNETIC WAVES

39.1. Conduction current results from the flow of positive charges towards negative charges and negative charges towards positive charges due to electrostatic attraction. This is seen to be consistent with the direction of flow shown in Fig. 39.1.

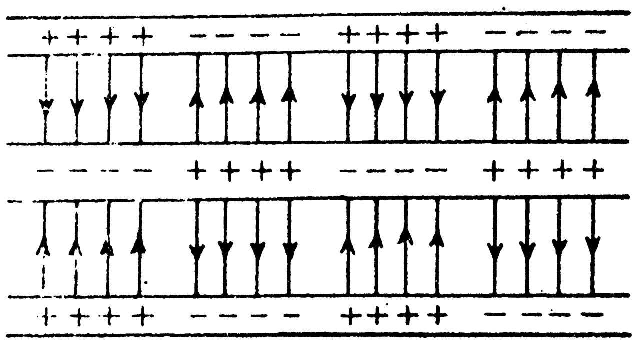


Fig. 39.1.

39.2. Assuming the dominant mode, with $a=3$ cm the as width of the rectangular waveguide, and $c=3\times10^8$ meter/sec, the velocity of electromagnetic waves in free space, the phase velocity is given by

$$v_{ph}=\frac{c}{\sqrt{1-(\lambda/2a)^2}}=\frac{3\times10^8 \text{ meter/sec}}{\sqrt{1-(\lambda \text{ cm}/6 \text{ cm})^2}}$$

where λ is the free space wavelength.

The group velocity is given by

$$v_{gr}=c\sqrt{1-(\lambda/2a)^2}=(3\times10^8 \text{ meter/see})\sqrt{1-(\lambda \text{ cm}/6 \text{ cm})^2}$$

and the guide wavelength is given by

$$\lambda_g=\frac{\lambda}{\sqrt{1-(\lambda/2a)^2}}=\frac{\lambda \text{ cm}}{\sqrt{1-(\lambda \text{ cm}/6 \text{ cm})^2}}.$$

Fig. 39.2. (*a*), (*b*) and (*c*) show the plot of v_{ph}, v_{gr} and λ_g as a function of λ.

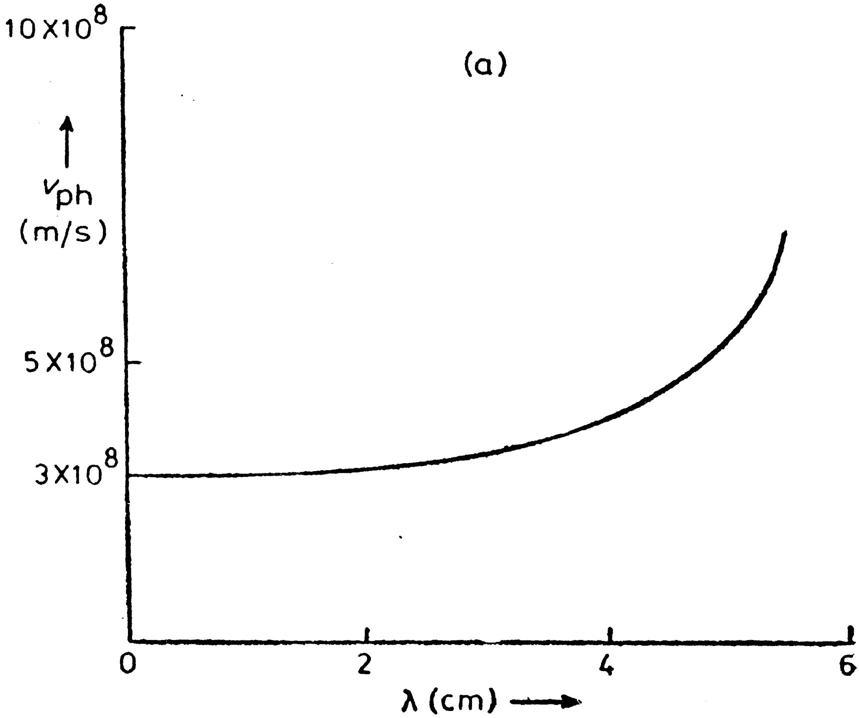


Fig 39.2. (a)

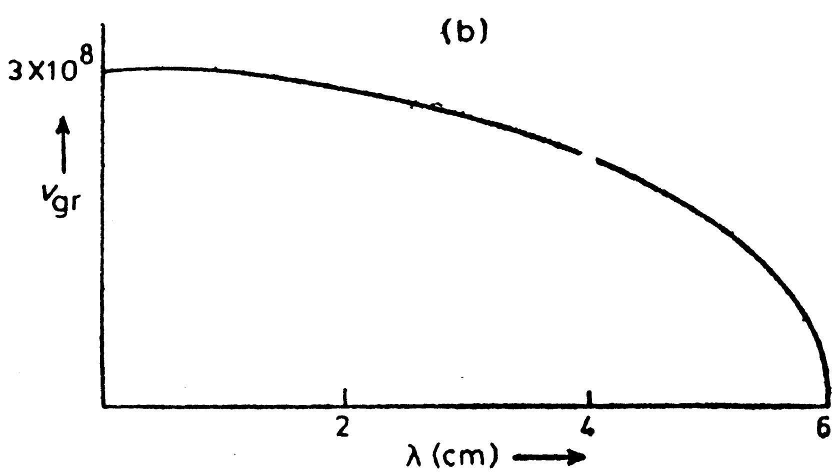


Fig 39.2. (b)

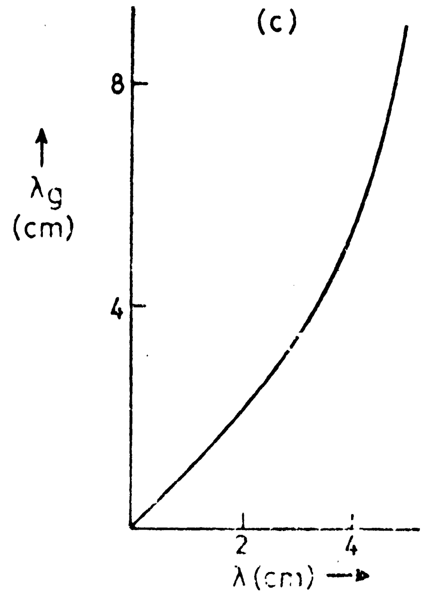


Fig 39.2. (c)

39.3. The group velocity v_{gr} is found from

$$v_{gr}=\frac{\text{distance}}{\text{time}}=\frac{100 \text{ meter}}{1.0\times 10^{-6} \text{ sec}}=10^8 \text{ meter/sec}$$

$$\therefore \quad \frac{v_{gr}}{c}=\frac{10^8 \text{ meter/sec}}{3\times 10^8 \text{ meter/sec}}=\frac{1}{3}$$

$$v_{gr}=c\sqrt{1-(\lambda/2a)^2}$$

Solving for λ,

$$\lambda=2a\sqrt{1-(v_{gr}/c)^2}=(2)(3 \text{ cm})\sqrt{1-(1/3)^2}=5.66 \text{ cm}.$$

Phase velocity, $v_{ph}=\dfrac{c^2}{v_{gr}}=\dfrac{(3\times 10^8 \text{ meter/sec})^2}{10^8 \text{ meter/sec}}=9\times 10^8$ meter/sec

39.4. Guide wavelength is given by

$$\lambda_g=\frac{\lambda}{\sqrt{1-(\lambda/2a)^2}} \qquad \ldots(1)$$

Set $\quad \lambda_g=2\lambda \qquad \ldots(2)$

Use (2) in (1) and solve for λ,

$$2\lambda=\frac{\lambda}{\sqrt{1-(\lambda/2a)^2}}$$

whence, $\quad \lambda=\sqrt{3}\, a.$

39.5. The displacement current i_d by definition is

$$i_d=\varepsilon_0\frac{d\phi_E}{dt}=\varepsilon_0\frac{d(EA)}{dt}=\varepsilon_0 A\frac{dE}{dt} \qquad ...(1)$$

where ϕ_E is the flux, E the electric field and A the area. The electric field for the plane wave is given by

$$E=E_m \sin(kx-\omega t) \qquad ...(2)$$

$$\therefore \quad \frac{dE}{dt}=-\omega E_m \cos(kx-\omega t) \qquad ...(3)$$

Using (3) in (1), and setting $A=1$

$$i_d=-\varepsilon_0\,\omega\,E_m \cos(kx-\omega t)$$

39.6. The density of energy stored in the electric field is

$$u_E=\tfrac{1}{2}\varepsilon_0 E^2 \qquad ...(1)$$

The density of energy stored in magnetic field is

$$u_B=\frac{1}{2\mu_0}B^2 \qquad ...(2)$$

The fields for the plane wave are

$$E=E_m \sin(kx-\omega t) \qquad ...(3)$$

$$B=B_m \sin(kx-\omega t) \qquad ...(4)$$

Substituting (3) in (1) and (4) in (2)

$$u_E=\tfrac{1}{2}\varepsilon_0 E_m^2 \sin^2(kx-\omega t) \qquad ...(5)$$

$$u_B=\frac{1}{2}\frac{B_m^2}{\mu_0}\sin^2(kx-\omega t) \qquad ...(6)$$

Dividing (5) by (6),

$$\frac{u_E}{u_B}=\frac{\varepsilon_0\mu_0 E^2_m}{B^2_m} \qquad ...(7)$$

But $\varepsilon_0\mu_0=\dfrac{1}{c^2}$ ...(8)

and $E_m=cB_m$ ...(9)

Using (8) and (9) in (7)

$$\frac{u_E}{u_B}=1 \quad \text{or } u_B=u_B$$

38.7. For the coaxial resonator of length L and closed at each end, the resonant frequencies are $\nu_n=nv/2L$.

By Problem, $L=3\lambda/2$, whence $n=3$.

$$E_r=\frac{1}{r}\sin\frac{n\pi z}{L}=\frac{1}{r}\sin\frac{3\pi z}{L}$$

$$B\phi=\frac{i}{cr}\cos\frac{n\pi z}{L}=\frac{i}{cr}\cos\frac{3\pi z}{L}$$

(See also Wave Guides by H.R.L. Lamont, John Wiley & Sons

39.8.

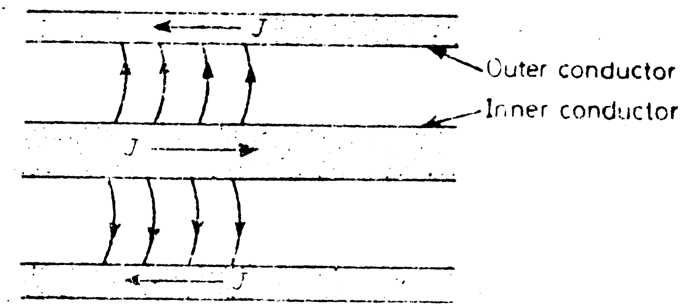


Fig. 39.8

39.9. The guide wavelength λ_g is given by

$$\lambda_g=\frac{\lambda}{\sqrt{1-\left(\frac{\lambda}{2a}\right)^2}}=\frac{10\text{ cm}}{\sqrt{1-\left(\frac{10\text{ cm}}{2\times 6\text{ cm}}\right)^2}}=18\text{ cm}$$

The cut-off wavelength λ_c for this guide in the fundamental mode is given by

$$\lambda_c=2a=(2)(6\text{ cm})=12\text{ cm}.$$

39.10. Figure 39.10 shows the electric field lines for an oscillating dipole at various stages of the cycle. In the beginning the electric lines will be attached to the charges but as the charges are reversed the electric lines are detached, forming closed loops. The magnetic field lines are concentric circles around the dipole axis. Far from the dipole the wave is essentially spherical moving radially outward with velocity equal to that of light.

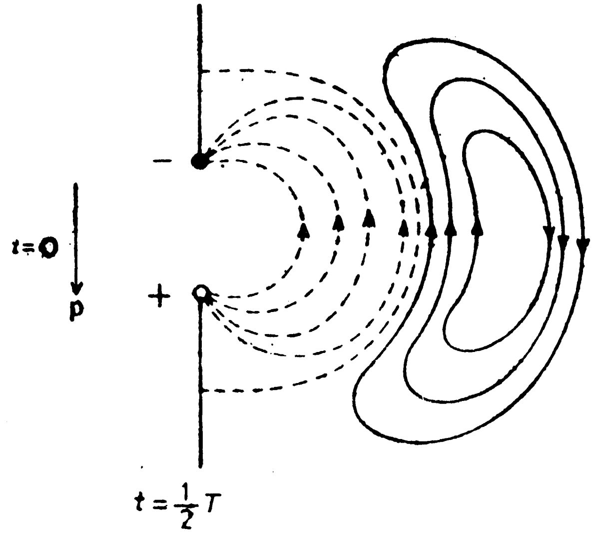


Fig 39. 10. (e)

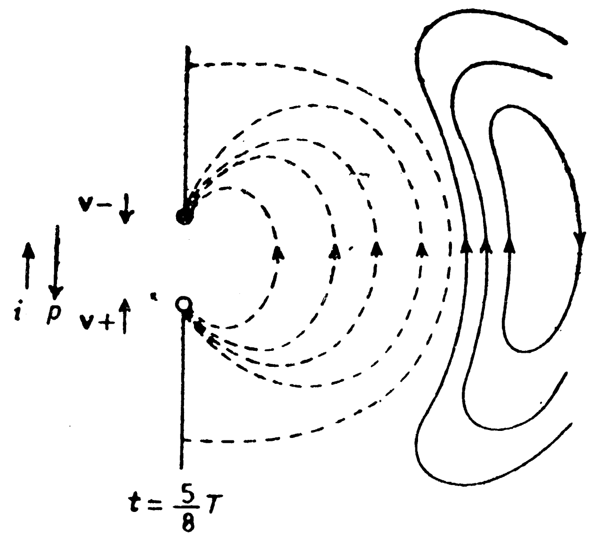


Fig. 39.10. (f)

Figure 39.10. shows four figures (*e*), (*f*), (*g*) and (*h*) in sequence to Textbook Fig. 39.9 (*a*), (*b*), (*c*) and (*d*). The fifth figure, Fig. 39.10 (*i*) is identical to Fig. 39.9 (*a*). The lines of **B** form closed loop that move away from the dipole with speed *c*.

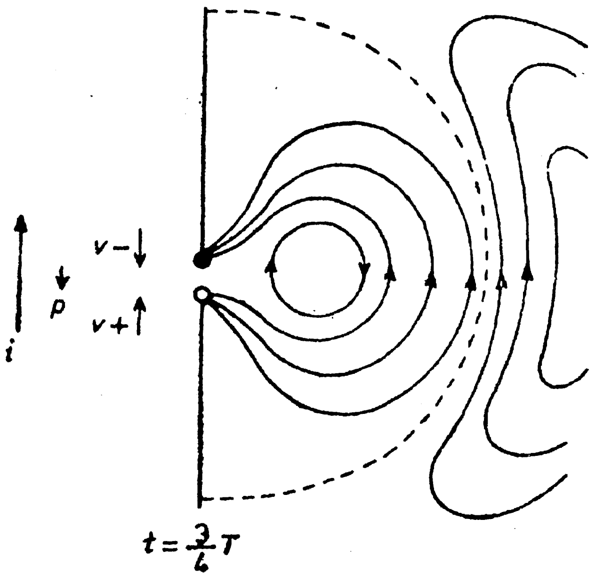


Fig. 39.10 (g)

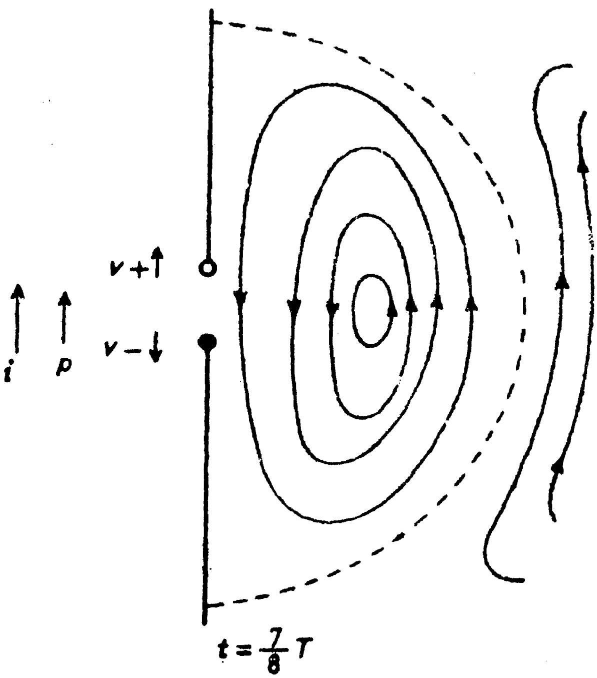


Fig 39.10 (h)

39.11. (*a*)

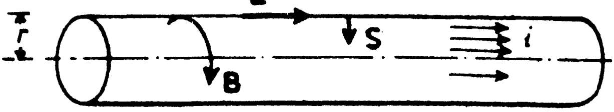


Fig. 39.11

(*b*) Consider a length L of the conductor whose return path is remote. Let the voltage drop be V. Then

$$E_z = \frac{V}{L} \quad \ldots(1)$$

$$B_\phi(r) = \frac{i}{2\pi r} \quad \ldots(2)$$

The Poynting vector is integrated over the surface of the cylinder. As S is directed radially inward, the caps at the ends do not make any contribution to $\oint \mathbf{S}.d\mathbf{A}$. The curved surface alones makes contribution to the integral. Thus.

$$\oint \mathbf{S}.d\mathbf{A} = \int (\mathbf{E}_Z \times \mathbf{B}_\phi).d\mathbf{A}$$

Observe that E_Z and B_ϕ are mutually perpendicular to each

$$\oint \mathbf{S}.d\mathbf{A} = \int_0^L E_Z B_\phi 2\pi r \, dz$$

$$= \frac{V}{L} \, i \int_0^L dz = Vi$$

where use has been made of (1) and (2). Now the quantity Vi represents ohmic loss. We, therefore, conclude that the loss is supplied by energy entering through the surface of the wire.

Within the conductors the Poynting vector is directed radially inward so that the ohmic loss equal to i^2R is supplied by the fields within the conductors. However, between the conductors the Poynting vector is directed towards the load so that the energy directed toward the load flows through the dielectric around the conductor and not through the conductor itself.

39.12. For the present problem,

$$E_x = E_{0x} \sin(\pi y/b) \exp[j(\omega t - kz)]$$

$$E_y = 0$$

$$E_z = 0$$

$$B_x = 0$$

$$B_y = \frac{kE_{0x}}{\omega} \sin(\pi y/b) \exp[j(\omega t - kz)]$$

$$B_z = \frac{\pi E_{oox}}{\omega b} \cos(\pi y/b) \exp[j(\omega t - kz)]$$

where $k = 2\pi/\lambda_g$ is the wave-number.

The average value of the Poynting vector is

$$\mathbf{S}_{av} = \frac{1}{2\mu_0} Re\,(\mathbf{E} \times \mathbf{B}^*) \qquad \ldots($$

where $\mathbf{B}^*$ is the complex conjugate of $\mathbf{B}$.

$$S_{av} = \frac{1}{2\mu_0} Re \begin{vmatrix} i & j & k \\ E_x & 0 & 0 \\ 0 & B_y^* & B_z^* \end{vmatrix}$$

$$= \frac{1}{2\mu_0} Re\,(-E_x B_z^* \,\mathbf{J} + E_x B_y^* \mathbf{K}) \qquad \ldots($$

Substituting the values of E_x, B_y and B_z in (2), we find that th first term in the parenthesis of (2) is imaginary whereas the secon term is real. The energy, therefore, flows only in the z-directio and,

$$S_{av} = \frac{kE^2_{oox}}{2\omega\mu_0} \sin^2(\pi y/b)\mathbf{K}$$

The value of S_{av} is independent of x and is zero at the wall ($y=0$ and $y=b$) where E is zero and is maximum at $y=b/2$.

39.13. (*a*) The rate at which the energy passes through a face o the cube is

$$P = \int \mathbf{S}.d\mathbf{A}$$

As $\mathbf{E}$ and $\mathbf{B}$ are in the xy plane, the Poynting vector $\mathbf{S}$ points in the z-direction, the direction of wave propagation. Thus, the energy passes only through faces parallel to xy plane. It is zero in the remaining faces paralled to yz and zx planes. For the xy plane,

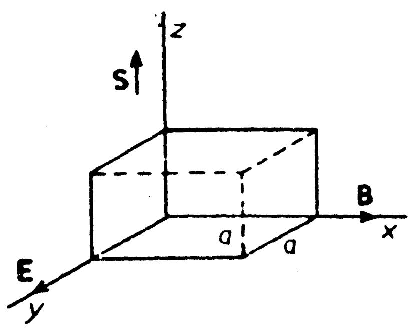


Fig. 39.13

$$P = S\int dA = SA$$

$$= A\frac{\mathbf{E}\times\mathbf{B}}{\mu_0} = \frac{AEB}{\mu_0} = \frac{a^2EB}{\mu_0}$$

where we have set $A = a^2$.

(*b*) As the energy flux within the cube is constant i.e. P is time independent, the net rate at which the energy in the cube changes is zero.

39.14. Radius of the wire, $r = 0.05 \times 2.54 \times 10^{-2}$ meter
$= 1.27 \times 10^{-3}$ meter.

Magnetic induction at distance r is

$$B = \frac{\mu_0 i}{2\pi r} = \frac{(4\pi \times 10^{-7}\ \text{weber/amp-m})(25\ \text{amp})}{(2\pi)(1.27 \times 10^{-3}\ \text{meter})}$$
$$= 3.9 \times 10^{-3}\ \text{weber/meter}^2.$$

The direction of **B** is tangent to the surface and perpendicular to the current.

Potential difference across 1000 ft or 305 meters is

$$V = iR = (25\ \text{amp})(0.1\ \text{ohm}) = 25\ \text{volts}$$

The electric field is

$$E = \frac{V}{d} = \frac{25\ \text{volts}}{305\ \text{meters}} = 8.2 \times 10^{-2}\ \text{volt/meter}$$

parallel to the current.

The Poynting vector is given by

$$\mathbf{S} = \frac{1}{\mu_0}\,\mathbf{E} \times \mathbf{B} = \frac{EB}{\mu_0}$$

since the angle between E and B is 90°.

$$\therefore \qquad S = \frac{(8.2 \times 10^{-2}\ \text{volt/meter})(3.9 \times 10^{-3}\ \text{weber/meter}^2)}{(4\pi \times 10^{-7}\ \text{weber/amp-m})}$$
$$= 255\ \text{watts/meter}^2.$$

The vector **S** is perpendicular to the plane defined by **E** and **B**.

It is directed radially inward, a result which is given from the rule of finding the resultant of the cross product of two vectors.

39.15. The average value of the Poynting vector is

$$\bar{S} = \frac{1}{2\mu_0} E_m B_m \qquad \ldots(1)$$

Also, $E_m = cB_m \qquad \ldots(2)$

$$\therefore \qquad \bar{S} = \frac{cB_m^2}{2\mu_0} \qquad \ldots(3)$$

Intensity of radiation $= \bar{S} = 2.0$ cal/cm²

$$= \frac{(2.0\ \text{cal})(4.2\ \text{joule})}{(1.0 \times 10^{-2}\ \text{meter})^2(60\ \text{sec})} = 1.4 \times 10^3\ \text{watts/meter}^2.$$

From (3) we have

$$B_m=\sqrt{\frac{2\bar{S}\mu_0}{c}}=\left[\frac{(2)(1.4\times10^2\text{ joule/m}^2)(4\pi\times10^{-7}\text{ weber/amp-m})}{(3\times10^8\text{ meter/sec})}\right]^{1/2}$$

$$=3.4\times10^{-6}\text{ weber/meter}^2.$$

Using this result in (2),

$$E_m=(3\times10^8\text{ meter/sec})(3.4\times10^{-6}\text{ weber/meter}^2)$$
$$=1020\text{ volt/meter.}$$

39.16. (*a*) $B_m=\frac{E_m}{c}=\frac{10^{-4}\text{ volt/meter}}{3\times10^8\text{ meter/sec}}=3.3\times10^{-13}\text{ weber/meter}^2.$

(*b*) $\bar{S}=\frac{cB_m^2}{2\mu_0}=\frac{(3\times10^8\text{ meter/sec})(3.3\times10^{-13}\text{ weber/meter}^2)^2}{(2)(4\pi\times10^{-7}\text{ weber/amp-m})}$

$$=1.3\times10^{-11}\text{ watts/meter}^2.$$

39.17. (*a*) **E** is parallel to the axis of the cylinder, in the direction of the current and **B** is tangential to the cylindrical surface.

Since, $\mathbf{S}=(\mathbf{E}\times\mathbf{B})/\mu$, it is directed inward normal to the surface following the rule for finding the resultant of the cross-product of two vectors. i.e. **S** is perpendicular to the plane formed by the vector **E** and **B**.

(*b*) The rate at which the energy flows into the resistor through the cylindrical surface is

$$P=\int \mathbf{S}.d\mathbf{A} \qquad \text{...(1)}$$

Since **S** is directed normal to the spherical surface, we need be concerned only with the surface area A of the cylinder.

$$A=2\pi al \qquad \text{...(2)}$$

$$\mathbf{S}=\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}=\frac{EB}{\mu_0} \qquad \text{...(3)}$$

where, we have used the fact that **E** and **B** are at right angles.

$$E=\frac{V}{l}=\frac{iR}{l} \qquad \text{...(4)}$$

At the surface, $B=\frac{\mu_0 i}{2\pi a}$ ...(5)

where we have assumed that $a \ll l$.

Integrating (1) over the closed surface,

$$P=S\int dA=SA=\frac{(EB)}{\mu_0}(2\pi rl)$$

$$=\frac{1}{\mu_0}\ \frac{iR}{l}\ \frac{\mu_0 i}{2\pi a}2\pi al=i^2R.$$

The quantity i^2R is nothing but the rate of Joule heating.

39.18. (*a*) **E** points in the direction of i, and **B** is tangential to the cylindrical surface. Hence, **S** which is given by the cross-product of the two vectors **E** and **B** points perpendicular to the plane containing **E** and **B** i.e. points radially inward.

(*b*) Let the radius of the parallel plate capacitor be r, then

$$a=\pi r^2 \qquad \cdots(1)$$

The induction at the surface of a cylinder of radius r is

$$B=\frac{1}{2}\ \mu_0\varepsilon_0 r\frac{dE}{dt} \qquad \cdots(2)$$

$$\mathbf{S}=\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}=\frac{EB}{\mu_0} \qquad \cdots(3)$$

where we have used the fact that **E** and **B** are perpendicular to each other.

Now,

$$P=\int \mathbf{S}.d\mathbf{A}=S\int dA=SA \qquad \cdots(4)$$

where we have used the fact that $d\mathbf{A}$ is normal to **S**, and that only the curved surface contributes to the integration.

$$A=2\pi rd \qquad \cdots(5)$$

Using (3) and (5) in (4),

$$P=\frac{EB}{\mu_0}(2\pi rd) \qquad \cdots(6)$$

Using (2) in (6)

$$P=\frac{E}{\mu_0}\tfrac{1}{2}\mu_0\varepsilon_0 r\ \frac{dE}{dt}(2\pi rd)$$

$$=\pi r^2 d\varepsilon_0 E\ \frac{dE}{dt}$$

or
$$P=\int \mathbf{S}.dA=ad\frac{d}{dt}(\tfrac{1}{2}\varepsilon_0E^2)$$

where we have used (1) and the identity

$$E\frac{dE}{dt}=\frac{1}{2}\frac{d}{dt}E^2.$$

SUPPLEMENTARY PROBLEMS

S.39.1. (*a*) Let l be the length of the coaxial cylindrical capacitor. (Textbook Fig. 30.4). Applying Gauss's law

$$\varepsilon_0 \oint \mathbf{E}.d\mathbf{S}=q \qquad ...(1)$$

we find

$$\varepsilon_0 E(2\pi rl)=q \qquad ...(2)$$

the flux being entirely through the cylindrical surface and zero through the end caps.

From (2) we have

$$E=\frac{q}{2\pi\varepsilon_0 rl} \qquad ...(3)$$

The potential difference between the plates is

$$V=\int_a^b Edr=\int_a^b \frac{q}{2\pi E_0 l}\frac{dr}{r}=\frac{q}{2\pi E_0 l}\ln\frac{b}{a}$$

Capacitance $C=\dfrac{q}{V}=\dfrac{2\pi\varepsilon_0 l}{\ln\dfrac{b}{a}}$

Hence, capacitance per unit length of the cable is

$$\frac{C}{l}=\frac{2\pi\varepsilon_0}{\ln\dfrac{b}{a}}$$

(*b*) The magnetic induction between the conductors is

$$B=\frac{\mu_0 i}{2\pi r} \qquad ...(4)$$

The energy density is

$$u=\frac{1}{2\mu_0}B^2=\frac{\mu_0 i^2}{8\pi^2 r^2} \qquad ...(5)$$

where use has been made of (4).

Consider a volume element dv for the cylindrical shell of radii r and $r+dr$ and of length b (Textbook Fig. 36-7). The energy contained in this volume element is

$$dU=udv=\frac{\mu_0 i^2}{8\pi^2 r^2}(2\pi rldr)=\frac{\mu_0 i^2 l}{4\pi}\frac{dr}{r} \qquad ...(6)$$

The total magnetic energy stored is given by integrating (6)

$$U=\int dU=\frac{\mu_0 i^2 l}{4\pi}\int_a^b \frac{dr}{r}=\frac{\mu_0 i^2 l}{4\pi}\ln\frac{b}{a} \qquad ...(7)$$

But $\quad U=\frac{1}{2}Li^2 \qquad ...(8)$

where L is the inductance.

Comparing (7) and (8),

$$L=\frac{\mu_0 l}{2\pi}\ln\frac{b}{a}$$

Hence, inductance per unit length of the cable is

$$\frac{L}{l}=\frac{\mu_0}{2\pi}\ln\frac{b}{a}$$

S.39.2. $\frac{\partial E}{\partial x}=-\frac{\partial B}{\partial t}$, Textbook Eq. 39.10.

$$-\frac{\partial B}{\partial x}=\mu_0\varepsilon_0\frac{\partial E}{\partial t}\text{, Textbook Eq. 39.13.}$$

Differentiating (39-10) with respect to x,

$$\frac{\partial^2 E}{\partial x^2}=-\frac{\partial^2 B}{\partial x\partial t}=\frac{\partial}{\partial t}\left(-\frac{\partial B}{\partial x}\right)=\frac{\partial}{\partial t}\left(\mu_0\varepsilon_0\frac{\partial E}{\partial t}\right)=\mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2}$$

where use has been made of (39-13).

In order to obtain the equation satisfied by B, differentiate (39-13) with respect to x

$$-\frac{\partial^2 B}{\partial x^2}=\mu_0\varepsilon_0\frac{\partial^2 E}{\partial x\partial t}=\mu_0\varepsilon_0\frac{\partial}{\partial t}\left(\frac{\partial E}{\partial x}\right)$$

$$=\mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\frac{\partial B}{\partial t}\right)=\mu_0\varepsilon_0\frac{\partial^2 B}{\partial t^2}$$

where use has been made of (39.10). We therefore obtain

$$\frac{\partial^2 B}{\partial x^2}=\mu_0\varepsilon_0\frac{\partial^2 B}{\partial t^2}$$

S.39.3. (*a*) $\frac{\partial E}{\partial x}=Ac$

$$-\frac{\partial B}{\partial t}=Ac$$

Thus E and B satisfy (39-10)

$$-\frac{\partial B}{\partial x}=-A$$

$$\mu_0\varepsilon_0 \ \frac{\partial E}{\partial t}=\mu_0\varepsilon_0(-AC^2)=--A$$

Since $c^2=\dfrac{1}{\mu_0\varepsilon_0}$

Thus E and B satisfy (39-13)

(*b*) $$\frac{\partial E}{\partial x}=Ac$$

$$-\frac{\partial B}{\partial t}=Ac$$

Thus, E and B satisfy (39-10).

$$-\frac{\partial B}{\partial x}=A$$

$$\mu_0\varepsilon_0 \ \frac{\partial E}{\partial t}=\frac{Ac^2}{c^2}=A$$

Thus E and B satisfy (39.1 3).

(*c*) $$\frac{\partial E}{\partial x}=Ac \ e^{x-ct}$$

$$-\frac{\partial B}{\partial t}=-(-Ace^{x-ct})=Ace^{x-ct}$$

Thus (39.10) is satisfied.

$$-\frac{\partial B}{\partial x}=-Ae^{x-ct}$$

$$\mu_0\varepsilon_0 \ \frac{\partial E}{\partial t}=\frac{1}{c^2}(-Ac^2)e^{x-ct}=-Ae^{x-ct}$$

Thus (39.13) is satisfied.

(*d*) $$\frac{\partial E}{\partial x}=\frac{Ac}{x+ct}$$

$$-\frac{\partial B}{\partial t}=\frac{Ac}{x+ct}.$$ Thus (39.10) is satisfied.

$$-\frac{\partial B}{\partial x}=\frac{A}{x+ct}$$

$$\mu_0\varepsilon_0 \frac{\partial E}{\partial t}=\frac{1}{c^2}\frac{Ac^2}{x+ct}=\frac{A}{x+ct}$$

Thus (39.13) is satisfied.

(*e*) To show that the given functions do satisfy equation (39.10) and (39.13) write for convenience

$$y=x-ct$$

Then, $\dfrac{\partial E}{\partial x}=Ac\dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial x}=Ac\dfrac{\partial f}{\partial y}$

as, $\dfrac{\partial y}{\partial x}=1.$

Also, $-\dfrac{\partial B}{\partial t}=-A\dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t}=Ac\dfrac{\partial f}{\partial y}$

as, $\dfrac{\partial y}{\partial t}=-c$

Thus (39.10) is satisfied.

$$-\frac{\delta B}{\delta x}=-A\frac{\delta f}{\delta y}\frac{\delta y}{\delta x}=-A\frac{\delta f}{\delta y}$$

$$\mu_0\varepsilon_0\frac{\delta E}{\delta t}=\frac{Ac}{c^2}\frac{\delta f}{\delta y}\frac{\delta y}{\delta t}=\frac{Ac}{c^2}\frac{\delta f}{\delta y}(-c)=-A\frac{\delta f}{\delta y}$$

Thus, (39.13) is satisfied.

The corresponding situation for functions of $(x+ct)$ is

$$E=Acf(x+ct)$$
$$B=-Af(x+ct)$$

S.39.4. (*a*) $E=E_m \sin \omega t \sin kx$ ...(1)

$B=B_m \cos \omega t \cos kx$ ...(2)

$$\frac{\delta E}{\delta x}=-\frac{\delta B}{\delta t} \quad ...(39\text{-}10)$$

$$-\frac{\delta B}{\delta x}=\mu_0 E_0\frac{\delta E}{\delta t}=\frac{1}{c^2}\frac{\delta E}{\delta t} \quad ...(39.13)$$

Differentiate (1) with respect to x and (2) with respect to t and use the resulting expressions in (39.10).

$$\frac{\delta E}{\delta x}=kE_m \sin \omega t \cos kx$$

$$\frac{\delta B}{\delta t}=-\omega B_m \sin \omega t \cos kx$$

$\therefore\ kE_m \sin \omega t \cos kx = \omega B_m \sin \omega t \cos kx$

or $\qquad kE_m = \omega B_m \qquad \ldots(3)$

Differentiate (2) with respect to x and (1) with respect to t and use the resulting expressions in (39.13).

$$\frac{\delta B}{\delta x} = -B_m k \cos \omega t \sin kx$$

$$\frac{\delta E}{\delta t}\ E_m \omega \cos \omega t \sin kx$$

$$\therefore\ B_m k \cos \omega t \sin kx = \frac{1}{c^2}\ E_m \omega \cos \omega t \sin kx$$

or $\qquad B_m k = \dfrac{E_m \omega}{c^2} \qquad \ldots(4)$

Multiply (3) and (4),

$$k^2 = \frac{\omega^2}{c^2}$$

or $\qquad \omega = ck \qquad \ldots(5)$

Use (4) in (5) to find,

$E_m = cB_m \qquad \ldots(6)$

(*b*) The Poynting vector is

$$\mathbf{S} = \frac{1}{\mu_0}\ \mathbf{E} \times \mathbf{B} = \frac{EB}{\mu_0} \qquad \ldots(7)$$

Since $\mathbf{E}$ and $\mathbf{B}$ are at right angles.

Use (1) and (2) in (7),

$$S = \frac{(E_m \sin \omega t \sin kx)(B_m \cos \omega t \cos kx)}{\mu_0}$$

$$= \frac{(E_m B_m)(2 \sin \omega t \cos \omega t)(2 \sin kx \cos kx)}{4\mu_0}$$

$$= \frac{E_m B_m}{4\mu_0} \sin 2\, kx \sin 2\, \omega t$$

(*c*) $$S_{av} = \frac{1}{T} \int_0^T S \sin \omega t\, dt$$

$$= \frac{1}{T} \int \frac{E_m B_m}{4\mu_0} \sin 2\, kx \sin 2\, \omega t \sin \omega t dt$$

$$=\frac{E_m B_m \sin 2kx}{2\mu_0 T}\int_0^T \sin^2 \omega t \cos \omega t dt$$

$$=\frac{E_m B_m \sin 2kx}{2\mu_0 T\omega}\int_0^T \sin^2 \omega t \, d(\sin \omega t)$$

$$=0$$

(*d*) The fact that $S_{av}=0$ means that the net flow of energy across a given area is zero. This is reasonable since we are here concerned with a standing wave.

S.39.5. (*a*) $S_{av}=\frac{1}{2}\frac{E_m^2}{\mu_0 c}$

or $\quad E_m=\sqrt{2\mu c S_{av}}$

$$=\sqrt{(2)(4\pi\times 10^{-7}\text{ weber/amp-m})(3\times 10^8\text{ meter/sec})(10\times 10^{-6}\text{ watt})}$$
$$=8.68\times 10^{-2}\text{ volt/meter.}$$

The effective or root-mean-square electric field is

$$E_{(rms)}=\frac{E_m}{\sqrt{2}}=6.14\times 10^{-2}\text{ volt/meter.}$$

(*b*) $\quad B_m=\frac{E_m}{c}=\frac{8.68\times 10^{-2}\text{ volt/meter}}{3\times 10^8\text{ meter/sec}}=2.9\times 10^{-10}\text{ weber/m}^2$

$$B_{(rms)}=\frac{B_m}{\sqrt{2}}=2.1\times 10^{-10}\text{ weber/m}^2.$$

(*c*) Total power radiated

$$P=4\pi r^2 S$$
$$=(4\pi)(10\times 10^3\text{ meter})^2(10\times 10^{-6}\text{ watts/meter}^2)$$
$$=1.26\times 10^4\text{ watts.}$$

S.39.6. (*a*) By Supplementary Problem 39.1, the potential difference between the plates is

$$V=\frac{q}{2\pi\varepsilon_0 l}\ln\frac{b}{a}=\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{b}{a} \qquad \ldots(1)$$

where $\quad \lambda=\frac{q}{l}$ is the charge density.

Let $\quad V=E$ in (1) and using the result of Problem 28.17,

$$E=\frac{\lambda}{2\pi\varepsilon_0 r}=\frac{E}{r\ln b/a} \qquad \ldots(2)$$

For $a < r < b$,

$$B = \frac{\mu_0 i}{2\pi r} = \mu_0 E/2\pi r R \qquad ...(3)$$

(*b*)

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$$

$$= \frac{1}{\mu_0} EB = \frac{1}{\mu_0}\,\frac{E}{r \ln b/a}\,\frac{\mu_0 E}{2\pi r R}$$

$$= \frac{E^2}{2\pi r^2 R \ln b/a} \qquad ...(4)$$

where use has been made of (2) and (3).

(*c*) The total power flowing across the annular section $a < r < b$ is given by

$$P = \int_a^b S(2\pi r dr) \qquad ...(5)$$

Using (4) in (5),

$$P = \int_a^b \frac{E^2(2\pi r dr)}{2\pi r^2 R \ln b/a}$$

$$= \frac{E^2}{R \ln b/a}\int_a^b \frac{dr}{r} = \frac{E^2}{R \ln b/a} \ln\frac{b}{a} = \frac{E^2}{R}.$$

Now, $E^2/R = P_J$, the rate of Joule heating. Therefore, the result obtained is reasonable. The Poynting theorem locates the flow of energy entirely in the field space between the conductors.

(*d*) If the battery is reversed, then $\mathbf{E} \to -\mathbf{E}$ and because current is reversed, $\mathbf{B} \to -\mathbf{B}$, and consequently

$$\mathbf{S} \to \frac{1}{\mu_0}(-\mathbf{E})\times(-\mathbf{B}) = \frac{\mathbf{E}\times\mathbf{B}}{\mu_0} = \mathbf{S}$$

Hence $\mathbf{S}$ is unchanged in magnitude as well as in direction.

S.39.7. (*a*) The charge on the surface is,

$$q = (2\pi R l)\,\sigma$$

current $\qquad i = qv = q\frac{\omega}{2\pi} = \frac{2\pi R l \sigma \omega}{2\pi} = R l \sigma \omega$

Magnetic induction

$$B=\frac{\mu_0 i}{l}=\frac{\mu_0(Rl\sigma\omega)}{l}=\mu_0\sigma\omega R\cdot \qquad \ldots(1)$$

(*b*) Faraday's law of induction can be written as

$$\oint \mathbf{E}_E.d\mathbf{l}=E=E_E(2\pi R)=\frac{-d\phi_B}{dt}$$

$$=-\frac{d}{dt}(\pi R^2 B)=-\pi R^2\frac{dB}{dt}$$

$$=-\pi R^2\mu_0\sigma\frac{Rd\omega}{dt}$$

where use has been made of (1)

$$E=-\frac{\pi R^3\mu_0\sigma\,\alpha}{2\pi R}=-\frac{1}{2}\mu_0\sigma R^2\alpha. \qquad \ldots(2)$$

(*c*) $$\mathbf{S}=\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}=\frac{1}{\mu_0}\left(-\frac{1}{2}\mu_0\sigma R^2\omega\right)(\mu_0\sigma R\omega)$$

$$=-\tfrac{1}{2}\mu_0\sigma^2R^3\omega\alpha. \qquad \ldots(3)$$

(*d*) Flux entering the interior volume of the cylinder

$$=SA=(\tfrac{1}{2}\mu_0\sigma^2R^3\omega L)(2\pi Rl)=\mu_0\pi\sigma^2R^4l\omega\alpha \qquad \ldots(4)$$

Now, $$\frac{dB}{dt}=\mu_0\tau R\frac{d\omega}{dt}=\mu_0\sigma R\alpha \qquad \ldots(5)$$

where use has been made of (1). Now (4) can be rewritten as

$$SA=\left(\frac{\pi R^2l}{\mu_0}\right)(\mu_0\sigma R\omega)(\mu_0\sigma R\alpha)=\frac{\pi R^2l}{\mu_0}B\frac{dB}{dt}=\frac{d}{dt}\left(\frac{\pi R^2lB^2}{2\mu_0}\right)$$

where use has been made of (1) and (5).

40 NATURE AND PROPAGATION OF LIGHT

40.1. From Textbook Fig. 40-2 we find the relative sensitivity of 50% for

(*a*) 510 $m\mu$ i.e. 5100 A° and 610 $m\mu$ i.e. 6100 A°

(*b*) Eye is most sensitive for $\lambda=5600$ A°.

$$\therefore \text{ Frequency, } \nu=\frac{c}{\lambda}=\frac{3\times10^{10}\text{ cm/sec}}{5600\times10^{-8}\text{ cm}}=5.4\times10^{14}\text{ c/s.}$$

$$\text{Time period, } T=\frac{1}{\nu}=\frac{1}{5.4\times10^{14}\text{ c/s}}=1.85\times10^{-15}\text{ secs.}$$

40.2. Gravitational force between sun and the space ship is

$$F=\frac{GMm}{r^2}$$

Mass of sun, $M=1.97\times10^{30}$ kg.

Mass of space ship, $m=100$ slugs$=100\times14.59$ kg$=1459$ kg.

$r=$sun-earth distance$=1.49\times10^{11}$ meters.

$G=6.67\times10^{-11}$ nt-m²/kg².

$$F=\frac{(6.67\times10^{-11}\text{ nt-m}^2/\text{kg}^2)(1.97\times10^{30}\text{ kg})(1459\text{ kg})}{(1.49\times10^{11}\text{ meter})^2}$$

$$\doteq 8.64\text{ nt.}$$

$$\text{Force/unit area}=\frac{2U}{c},$$

since it is assumed that the sail is perfect reflector.

$U=1400$ watts/meter²

$c=3\times10^{8}$ meter/sec

$$\text{Total force}=\frac{2U}{c}A,$$

where A is the area.

$$\text{Set } \frac{2U}{c}A=8.64\text{ nt/m}^2$$

or $$A=\frac{8.64\,c}{2U}=\frac{8.64\times3\times10^{8}}{2\times1400}$$
$$=9.2\times10^{5}\text{ meter}^2.$$

40.3. Under the assumption that earth completely absorbs radiation, force on earth due to radiation pressure is

$$F=\frac{U}{c}A$$

where $A=\pi R^2$, area of a flat disc whose radius is the radius of earth. With

$$U=1400 \text{ watts/meter}^2$$

$$c=3\times 10^8 \text{ meter/sec}$$

$$R=6.4\times 10^6 \text{ meter}$$

$$F=\frac{(1400 \text{ watts/meter}^2)}{(3\times 10^8 \text{ m/s})}(6.4\times 10^6 \text{ m})^2\,\pi=6.0\times 10^8 \text{ nt.}$$

Gravitational attraction between earth and sun

$$F=\frac{GMm}{r^2}$$

With $M=1.97\times 10^{30}$ kg

$m=6.0\times 10^{24}$ kg

$r=1.49\times 10^{11}$ meters

$G=6.67\times 10^{-11}$ nt-m^2/kg^2

$$F=\frac{(6.67\times 10^{-11} \text{ nt-m}^2/\text{kg}^2)(1.97\times 10^{30} \text{ kg})(6.0\times 10^{24} \text{ kg})}{(1.49\times 10^{11} \text{ m})^2}$$

$$=3.6\times 10^{22} \text{ nt.}$$

40.4. Let a plane electromagnetic wave fall perpendicularly on a perfectly absorbing surface. If the incident momentum per unit volume is p then the amount of momentum associated with the radiation falliag per unit time is given by the quantity pcA. By our assumption, the radiation is completely absorbed by the surface so that pcA is also the momentum absorbed per unit time by the surface of area A. Thus, the force on this area is

$$F=pcA.$$

Hence, the radiation pressure is given by

$$P_{(rad)}=\frac{F}{A}=cp=u=\varepsilon_0 E^2$$

where u is the energy density.

On the other hand, for a perfect reflector, the reflected light has momentum equal in magnitude but opposite in direction so that the change in momentum per unit volume is 2 p and the corresponding radiation pressure is

$$P_{(rad)}=2cp=2u=2\varepsilon^0 E^2$$

As the radiation pressure is doubled, the energy density is also doubled, half of the photons traversing towards the mirror and an equal number in the opposite direction after reflection. In general the above result holds irrespective of the incident energy that is reflected.

40.5. Let n bullets travelling with speed v strike a plane surface of area/meter^2 at right angles in unit time. Then,

$$\text{Pressure} = \frac{\text{force}}{\text{unit area}} = \text{momentum delivered/sec/unit/area}$$

$$= n \times mv \qquad \ldots(1)$$

under the assumption that bullets are completely absorbed by the surface.

Consider a box of length v meters and unit cross-section so that volume of the box is v meter^3.

Energy flowing/unit area/sec $= n \times \frac{1}{2} mv^2$.

Energy density $= n \times \frac{1}{2} mv^2/v = \frac{1}{2} nmv$. $\qquad \ldots(2)$

Compare (1) with (2) to conclude that 'pressure' is twice the energy density in the stream above the surface.

40.6. $m = 100$ slugs $= 14.59 \times 100$ kg $= 1459$ kg.

Power $= 10^4$ watts

$U =$ energy radiated in 1 day $= (10^4$ watts$)(86400$ secs$)$

$= 8.64 \times 10^8$ joules.

Momentum delivered to the space ship is,

$$p = \frac{U}{c} = \frac{(8.64 \times 10^8 \text{ joules})}{(3 \times 10^8 \text{ meter/sec})} = 2.88 \text{ nt/meter}^2.$$

$$\therefore \text{ Velocity increase} = \frac{p}{m} = \frac{2.88 \text{ nt/meter}^2}{(1459 \text{ kg})}$$

$$= 0.00197 \text{ meter/sec} \simeq 2.0 \text{ m/sec.}$$

10.7. Consider a sphere of radius $r = 1.0$ meter. Let the bulb be located at the centre of the sphere. Consider a small area A at distance r. Energy falling/sec over area $A = 500 \times \dfrac{A}{4\pi r^2}$ watts.

Set $\quad A = 1.0$ meter^2 and $r = 1.0$ meter.

$$U = \frac{500}{4\pi} = 39.8 \text{ watts/meter}^2.$$

Momendum delivered $P=\frac{U}{c}$

(Under the assumption that the surface is perfectly absorbing)

$$\text{Force}=\frac{p}{t}=\frac{U}{ct}=\frac{(39.8)}{(3\times10^{8})(1.0)}=1.3\times10^{-7}\text{ nt.}$$

$$\text{Radiative pressure}=\frac{\text{force}}{\text{area}}=\frac{1.3\times10^{-7}\text{ nt}}{1.0\text{ meter}^2}=1.3\times10^{-7}\text{ nt/meter}^2.$$

40.8. The uncertainty in the distance to the moon.

$$\delta R=0.5\text{ mile}=0.8\text{ km}=800\text{ meters.}$$

Therefore, uncertainty in time,

$$\delta t=\frac{\delta R}{c}=\frac{800\text{ meter}}{3\times10^{8}\text{ meter/sec}}=2.7\times10^{-6}\text{ sec}=2.7\ \mu\text{sec.}$$

40.9. Roemer (1673) was the first person to establish that light travels with finite speed at about 186,000 miles per sec. The planet Jupiter which revolves around the sun once in about 12 years, has 12 satellites of which four can be seen with the aid of a low power telescope. These satellites revolve around Jupiter in the same plane as Jupiter itself and each is eclipsed once in every revolution as it enters the shadow of Jupiter. The principle of the determination of velocity of light consists of estimating the apparent time of revolution of one of the satellites obtained from successive eclipses when the earth is nearest to Jupiter and when it is farthest. Roemer noticed that eclipses take place 16½ minutes or 1000 secs late when earth is farthest from Jupiter compared to when the earth is closest. The delay is attributed to the extra time that light has to take in travelling across the diameter of earth's orbit. Now, the earth's orbit was known from the method of triangulation and is around 1.86×10^8 miles. It follows that velocity of light,

$$c=\frac{1.86\times10^{8}}{1000}=186{,}000\text{ miles/sec.}$$

(*a*) Refering to Textbook Fig. 40.11 when earth moves from x to y, the earth-Jupiter distance has increased, Therefore, one expects the apparent time of revolution of Jupiter's setellite to increase.

(*b*) Two observations are needed.

(*i*) Timing of eclipses.

(*ii*) Diameter of earth's orbit around the sun.

40.10. The uncertainty of measurement in c is less than 0.0001%. We should not have an error in the length more than

$$\frac{\Delta R}{R}=\frac{\Delta c}{c}=\frac{0.0001}{100}=10^{-6}$$

Set, $R=1.0$ miles$=63360$ in

$\Delta R=(63360 \text{ in.})\times 10^{-6}=0.063$ inches.

40.11. Set $\dfrac{1}{\sqrt{1-v^2/c^2}}=1.01$

$$1-\frac{v^2}{c^2}=\left(\frac{1}{1.01}\right)^2=0.98$$

whence, $v/c=\sqrt{0.02}=014.$

40.12. $\lambda=4340\times 10^{-8}$ cm

$\lambda'=6562\times 10^{-8}$ cm

$\theta=180°$

(*a*) $$\nu'=\frac{\nu}{\sqrt{1-v^2/c^2}}\left(1-\frac{v}{c}\right)=\nu\sqrt{\frac{1-v/c}{1+v/c}}$$

$$\frac{1-v/c}{1+v/c}=\left(\frac{\nu'}{\nu}\right)^2=\left[\frac{c/\lambda'}{c/\lambda}\right]^2=\left(\frac{\lambda}{\lambda'}\right)^2$$

$$=\frac{(4340\times 10^{-8}\text{ cm})^2}{(6562\times 10^{-8}\text{ cm})^2}=0.437$$

whence, $v/c=0.39$

$\therefore$ $v=0.39\times 3\times 10^8=1.2\times 10^8$ meter/sec.

(*b*) It is receding.

40.13. (*a*) Let ν be the frequency of the incident microwave beam. Let the car be approaching with spped v. Then the frequency seen by the car is

$$\nu'=\nu\,(1+v/c) \qquad ...(1)$$

Upon reflection the microwave comes back as if it was emitted by a moving source travelling with speed v towards the observer. Therefore, the frequency observed

$$\nu''=\nu'\,(1+v/c)=\nu(1+v/c)^2 \qquad ...(2)$$

where we have used (1).

$$\Delta\nu=\nu''-\nu=\nu(1+v/c)^2-\nu$$

We can assume $v/c \ll 1$

$$\therefore \quad \Delta\nu=\nu\left(1+\frac{2v}{c}+\frac{v^2}{c^2}\right)-\nu$$

Neglect the quadratic term v^2/c^2.

$$\Delta\nu \simeq 2\nu \frac{v}{c}$$

For a receding car it can be shown, proceeding along similar lines, that $\Delta\nu \simeq -2\nu\, v/c$.

(*b*) $\Delta\nu = 2\nu \dfrac{v}{c}$

$$\nu = 2450 \text{ mega-cycles/sec} = 2450 \times 10^6 \text{ c/s.}$$

Let v be expressed in miles/hour.

$$c = 186{,}000 \text{ miles/sec} = 1.86 \times 10^5 \times 3600 \text{ miles/hr.}$$
$$= 6.7 \times 10^8 \text{ mph}$$
$$\frac{\Delta\nu}{v} = \frac{2\nu}{c} = \frac{2 \times 2450 \times 10^6 \text{ c/s}}{6.7 \times 10^8 \text{ mph}} = 7.3 \text{ (cycles/sec).}$$

40.14. $\nu' = \nu\,(1 - v/c)$ ...(1)

$$\Delta\lambda = \lambda' - \lambda$$
$$\therefore \quad \frac{\Delta\lambda}{\lambda} = \frac{\lambda'}{\lambda} - 1 = \frac{\nu}{\nu'} - 1$$
$$= \frac{1}{(1 - v/c)} - 1 = (1 - v/c)^{-1} - 1$$
$$= \left(1 + \frac{v}{c} \cdots\cdots\right) - 1 \simeq \frac{v}{c}$$

where we have neglected higher order terms in the series expansion.

40.15. $\lambda = 5500 \times 10^{-8}$ cm

Radius, $r = 7 \times 10^8$ meter $= 7 \times 10^{10}$ cm

Period of rotation, $T = 24.7$ days $= 24.7 \times 86400$ sec

$$= 2.13 \times 10^6 \text{ sec}$$

$$V = \omega r = 2\pi r/T$$

By Problem 40.14, $\Delta\lambda = \lambda \dfrac{v}{c} = \lambda \dfrac{2\pi}{T} \dfrac{r}{c}$

$$= \frac{(5500 \times 10^{-8} \text{ cm})(2\pi)(7 \times 10^{10} \text{ cm})}{(2.13 \times 10^6 \text{ sec})(3 \times 10^{10} \text{ cm/sec})}$$
$$= 3.8 \times 10^{-10} \text{ cm}$$
$$= 0.038 \text{ A}^\circ.$$

40.16. $\nu = 40 \times 10^6$ c/s

$$\nu' = \nu\,\left(1 + \frac{v}{c} \cos\theta\right)$$

$$\Delta\nu = \nu' - \nu = \nu = \frac{v}{c} \cos\theta.$$

Now, $AB=-vt$, since on left side of B, time is minus.

$BC=250$ miles.

$$\therefore \quad \cos\theta=\frac{AB}{AC}=\frac{-vt}{\sqrt{v^2t^2+(250)^2}}$$

where t is negative and is expressed in hours. For the path AB,

$$\therefore \quad \Delta\nu=\frac{-\nu v^2 t}{c\sqrt{v^2t^2+(250)^2}}=\frac{-3.87\times10^6\, t}{\sqrt{t^2+1.9\times10^{-4}}} \text{ (For the path } AB)$$

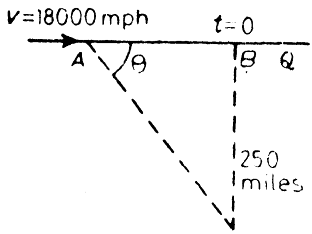


Fig 40.16 (a)

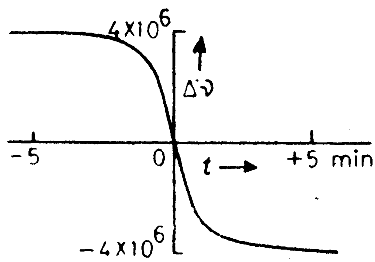


Fig. 40.16 (b)

For the path BQ, t is positive and consequently $\Delta\nu$ will be negative. Fig. 40.16 (b) shows the plot of $\Delta\nu$ against t.

40.17. $v/c=0.2$

$\lambda_{Blue}=4750\times10^{-8}$ cm

$$\nu'=\frac{\nu(1-v/c)}{\sqrt{1-v^2/c^2}}=\nu\sqrt{\frac{1-v/c}{1+v/c}}$$

$$\frac{\nu'}{\nu}=\sqrt{\frac{1-0.2}{1+0.2}}=0.816$$

$$\lambda'\nu'=\lambda\nu$$

$$\lambda'=\lambda\frac{\nu}{\nu'}=\frac{4750\times10^{-8}\text{ cm}}{0.816}=5821\times10^{-8}\text{ cm.}$$

The colour would be yellow-orange (see Textbook Fig. 40,2).

40.18. $$\beta=\frac{v}{c}=\frac{8.61\times10^5\text{ meter/sec}}{3.0\times10^8\text{ meter/sec}}=2.87\times10^{-3}$$

(a) $$\nu'=\nu\frac{(1+\beta\cos\theta)}{\sqrt{1-\beta^2}}$$

Set $\theta=180°$

$$\nu_1'=\nu\,(1-\beta)(1-\beta^2)^{-1/2}=\nu(1-\beta)(1+\tfrac{1}{2}\beta^2+\ldots\ldots)$$
$$=\nu(1-\beta+\tfrac{1}{2}\beta^2)$$
$$\Delta\nu_1=\nu-\nu_1'=\nu(\beta-\tfrac{1}{2}\beta^2)$$

Set $\theta=0$

$$\nu_2'=\nu(1+\beta)(1-\beta^2)^{-1/2}=\nu(1+\beta)(1+\tfrac{1}{2}\beta^2+\ldots\ldots)$$
$$=\nu(1+\beta+\beta=\tfrac{1}{2}\beta^2)$$
$$\Delta\nu_2=\nu_2'-\nu=(\beta+\tfrac{1}{2}\beta^2)$$
$$\frac{\Delta\nu}{\nu}=\frac{\Delta\nu_2-\Delta\nu_1}{\nu}=\beta^2=(2.87\times10^{-3})^2=0.824\times10^{-5}$$

(*b*) $$\nu_1'=\nu(1-\beta+\beta^2)$$
$$\Delta\nu_1=\nu(\beta-\beta^2)$$
$$\nu_2'=\nu(1+\beta+\beta^2)$$
$$\Delta\nu_2=\nu(\beta+\beta^2)$$
$$\frac{\Delta\nu}{\nu}=\frac{\Delta\nu_2-\Delta\nu_1}{\nu}=2\beta^2=2(2.87\times10^{-3})=1.65\times10^{-5}$$

See Textbook Table 40.2 for comparison.

SUPPLEMENTARY PROBLEMS

S.40.1. (*a*) $$\nu=\frac{c}{\lambda}=\frac{3\times10^8\text{ meter/sec}}{3\text{ meter}}=10^8\text{ cycles/sec}$$

(*b*) $$B_m=\frac{E_m}{c}=\frac{300\text{ volts/meter}}{3\times10^8\text{ meter/sec}}=10^6\text{ weber/m}^2$$

B is perpendicular to both **E** and the direction of propagation.

(*c*) $$E=E_m\sin(kx-\omega t)$$
$$k=\frac{2\pi}{\lambda}=\frac{2\pi}{3.0\text{ meter}}=2.09\text{ meter}^{-1}.$$
$$\omega=2\pi\nu=(2\pi)(10^8\text{ sec}^{-1})=6.28\times10^8\text{ radians/sec.}$$

(*d*) Time average rate of flow of energy is given by the average value of Poynting vector, S_{av},

$$S_{av}=\frac{1}{2\mu_0}E_mB_m=\frac{(300\text{ volts/meter})(10^{-6}\text{ weber/m}^2)}{(2)(4\pi\times10^{-7}\text{ weber/amp-m})}$$
$$=119.4\text{ watts/meter}^2.$$

(*e*) For a perfectly absorbing surface of area A momentum delivered per sec,

$$\frac{S_{av}}{c}=\frac{119.4 \text{ watts/meter}^2}{3\times10^8 \text{ meter/sec}}=4\times10^{-7}A \text{ nt.}$$

$$\text{Radiation pressure}=\frac{\text{force}}{\text{area}}=\frac{\text{momentum delivered/sec}}{\text{area}}$$

$$=\frac{4\times10^{-7}A \text{ nt}}{A}=4\times10^{-7} \text{ nt/meter}^2.$$

S.40.2. $\varepsilon_0\mathbf{E}\times\mathbf{B}=\varepsilon_0\mu_0\dfrac{\mathbf{E}\times\mathbf{B}}{\mu_0}=\dfrac{S}{c^2}$

Since $c^2=\dfrac{1}{\mu_0\varepsilon_0}$ and $\mathbf{S}=\dfrac{\mathbf{E}\times\mathbf{B}}{\mu_0}$ is the Poynting vector

$$[\varepsilon_0\mathbf{E}\times\mathbf{B}]=\frac{[S]}{[c^2]}=\frac{[\text{power/area}]}{[(\text{velocity})^2]}$$

$$=\frac{[ML^2T^{-3}/L^2]}{[LT^{-1}]^2}=\frac{[MLT^{-1}]}{[L^3]}=\frac{[\text{momentum}]}{[\text{volume}]}.$$

S.40.3. Gravitational force on the particle of mass m and density ρ at a distance r from the sun of mass M is

$$F_{gr}=G\frac{Mm}{r^2}=\frac{GM}{r^2}\,\frac{4\pi R^3\rho}{3}$$

$$=\frac{4\pi}{3}(6.67\times10^{-11} \text{ nt-meter}^2/\text{kg}^2)(1.98\times10^{30} \text{ kg})\left(1000\frac{\text{kg}}{m^3}\right)\frac{R^3}{r^2}$$

$$=55.3\times10^{22}\,\frac{R^3}{r^2} \text{ nt} \qquad \ldots(1)$$

where R is the particle radius.

Solar radiation received per second by the projected area πR^2 at distance r from the sun is

$$U=(\text{Mean total solar radiation})\,\frac{(\pi R^2)}{4\pi r^2}$$

$$=(3.92\times10^{26} \text{ watts})\,\frac{R^2}{4r^2}=9.8\times10^{25}\,\frac{R^2}{r^2} \text{ watts.}$$

Momentum delivered to the particle per second is,

$$p=\frac{U}{c}$$

Thus the radiation force

$$F_{rad}=\frac{U}{c}=\left(9.8\times10^{25}\,\frac{R^2}{r^2} \text{ watts}\right)\left(\frac{1}{3\times10^8 \text{ meter/sec}}\right)$$

Condition that the particle is blown out of the solar system is

$F_{gr} < F_{rad}$

Using (1) and (2) in (3),

$$55.3 \times 10^{22} \frac{R^3}{r^2} < 3.3 \times 10^{17} \frac{R^2}{r^2}$$

(*a*) or $R < 5.9 \times 10^{-7}$ meter ...(4)

(*b*) Set $R = R_0$ in (4) : The critical radius
$R_0 = 5.9 \times 10^{-7}$ meter.

(*c*) Condition (4) is independent of r. Hence, R_0 does not depend on r.

41 REFLECTION AND REFRACTION— PLANE WAVES AND PLANE SURFACES

41.1. A stationary disturbance produces spherical wave-fronts. However, in the present problem the wave-front of the disturbance can be found out by exciting in sequence a row of stationary sources of disturbance. We can determine the disturbance at a subsequent time t_2 by drawing spheres of radii $u\ (t_2-t_1)$ around various points along the path of the object. The resulting wave-front is the envelope of spherical waves which originate at the nose of the fast moving object at successive instants in its flight (Fig. 41.1). The wave-front is a common tangent to the secondary wavelets given by the usual Huygen's construction. The envelope would be,a cone of semi-angle α given by

$$\sin\alpha=\frac{ut}{vt}=\frac{u}{v}$$

[Should v be less then u than α would be imaginary and the spherical waves have no envelope. In other words, no wave front is formed.

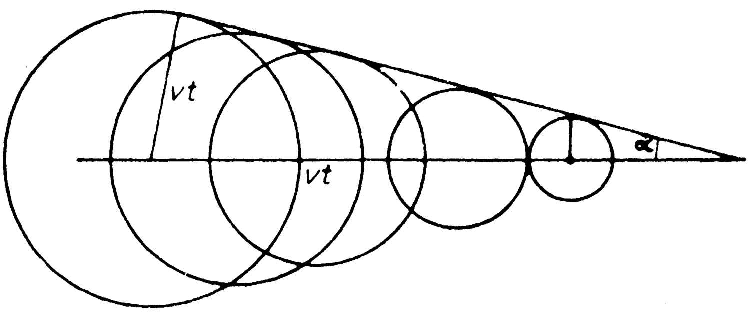


Fig. 41.1

41.2. Mirror is rotated through angle α. The normal is also rotated through angle α, $N\hat{B}N'=\alpha$. The reflected ray BC becomes BD for the new position so that angle of rotation of the reflected ray is $C\hat{B}D$.

$$A\hat{B}N'=N'\hat{B}D$$

$$C\hat{B}D=N'\hat{B}D+C\hat{B}N'$$

$$=A\hat{B}N'+(N\hat{B}N'-N\hat{B}C)$$

$$=N\hat{B}N'+A\hat{B}N+N\hat{B}N'-NBC$$

But $N\hat{B}N'=\alpha$ and $A\hat{B}N=N\hat{B}C$

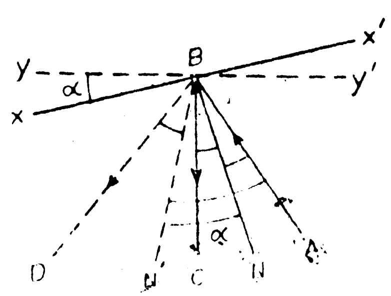


Fig. 41.2

$\therefore \quad C\acute{B}\dot{D}=2\alpha$

that is, the reflected ray is rotated through 2α if the plane mirror is rotated through α.

41.3. From Textbook Fig. 41.2, $n=1.46$ for $\lambda=5500$ A

$$v=\frac{c}{n}=\frac{3\times 10^8 \text{ meter/sec}}{1.46}=2.05\times 10^8 \text{ meter/sec}.$$

41.4. (*a*) $\nu=\dfrac{c}{\lambda}=\dfrac{3\times 10^{10}\text{ cm/sec}}{5890\times 10^{-8}\text{ cm}}=5.09\times 10^{14}\text{ sec}^{-1}.$

(*b*) $\lambda'=\dfrac{\lambda}{n}=\dfrac{5890\text{ A}}{1.52}=3875\text{ A}.$

(*c*) As the frequency in the medium remains unchanged

$$v=\nu\lambda'=(5.09\times 10^{14}\text{ sec}^{-1})(3875\times 10^{-8}\text{ cm})$$
$$=1.97\times 10^{10}\text{ cm/sec.}$$

41.5. $n=\dfrac{c}{v}=\dfrac{3\times 10^8\text{ meter/sec}}{1.92\times 10^8\text{ meter/sec}}=1.56$

41.6. (*a*) $\delta n=0.00001$

$$c=3\times 10^5\text{ km/sec}$$

$$n=\frac{c}{v}$$

or $\qquad c=nv$

$$dc=\frac{\partial c}{\partial n}dn+\frac{\partial c}{\partial v}dv$$

$$dc=v\,dn+n\,dv \qquad \ldots(1)$$

Since n and v are uncorrelated,

$$\delta c=\sqrt{\overline{(dc)^2}}=\sqrt{(\delta n)^2\, v^2+(\delta v)^2\, n^2} \qquad \ldots(2)$$

From (1) the extra uncertainty arising from uncertainty in refractive index is

$$v\delta n=\frac{c}{n}\delta n=(3\times 10^5\text{ km/sec})\left(\frac{0.00001}{1.0003}\right)=3\text{ km/sec}$$

(*b*) Now $n\delta v=(1.0003)(1\text{ km/sec})=1.0\text{ km/sec}$

$$\therefore \qquad \delta c=\sqrt{(3)^2+(1)^2}=\sqrt{10}=3.16\text{ km.}$$

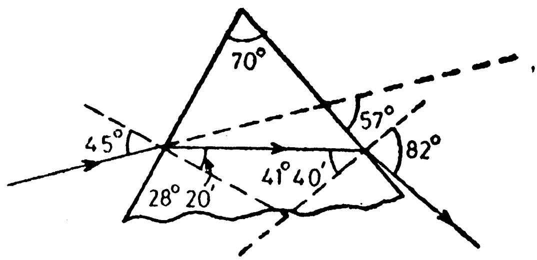
If we want to reduce δc to 1.1 km

Set $\quad \delta c = 1.1 = \sqrt{(3\times 10^5 \times \delta n)^2 + 1^2}$

or $\quad \delta n = 0.0000015.$

Thus, n should be measured with an accuracy better than a factor of 15.

41 7. In Textbook Fig. 41.6 we find the angle of incidence $\theta = 59°$ and angle of refraction $\alpha = 35°$, so that the refractive index $n = \frac{\sin\theta}{\sin\alpha} = \frac{\sin 59°}{\sin 35°} = 1.49$. The angle of deviation $\psi = 48°$.



Fig, 41.7 (a)

In Fig 41.7. (*a*), the ray is traced for the angle of incidence $\theta_1 = 45°$.

The angle of refraction α_1 at the first refracting face is calculated from $\sin\alpha_1 = \frac{\sin\theta_1}{n} = \frac{\sin 45°}{1.49}$. We find $\alpha_1 = 28°20'$.

As the angle of prism $\phi = 70°$, we find $\alpha_2 = \phi - \alpha_1 = 70° - 28°20' = 41°40'$. The angle of emergence θ_2 is calulated from $\sin\theta_2 = n \sin\alpha_2 = 1.49 \sin 41°40'$; whence $\theta_2 = 82°$. The angle of deviation $\psi = \theta_1 - \alpha_1 + \theta_2 - \alpha_2 = 45° - 28°20' + 82° - 41°40' = 57°$.

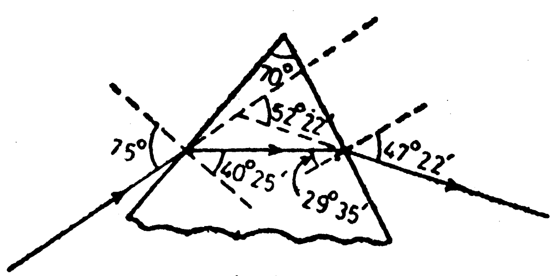


Fig. 41.7 (b)

In Fig. 41.7 (*b*), the ray is traced for the angle of incidence $\theta_1 = 75°$. Proceeding as above we find $\alpha_1 = 40°25'$, $\alpha_2 = 29°35'$, $\theta_2 = 47°22'$ and $\psi = 52°22'$.

41.8. (*a*) $n=1.52$

Critical angle of reflection,

$$\sin\theta_c=\frac{1}{n}=\frac{1}{1.52}=0.6579$$

$$\therefore \quad \theta_c=41.2^\circ \simeq 41^\circ$$

When ϕ is maximum, α will be least (Fig 41.8).

But α is complimentary to β and so also θ_1 is complimantary to β, so that $\alpha=\theta_1=\theta_2$. But least value of θ_2 is θ_c for which total internal reflection takes place.

$\therefore$ Max. value of ϕ is $90-\alpha=90-41^\circ=49^\circ$

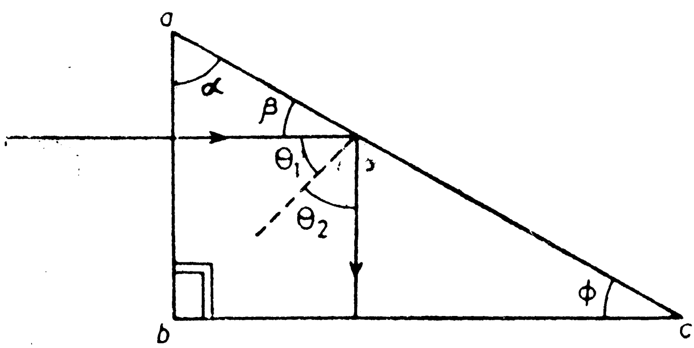


Fig. 41.8.

(*b*) $$n_{(water-glass)}=\frac{n_{glass}}{n_{water}}=\frac{1.52}{1.33}=1.14$$

$$\sin\theta_c=\frac{1}{n}=\frac{1}{1.14}=0.877$$

$$\theta_c=61.3^\circ\simeq 61^\circ$$

Set $$\alpha=61^\circ$$

$$\phi_{max}=90-\alpha=90-61=29$$

41.9. θ is small, so that α is also small. Hence $\psi=2(\theta-\alpha)$ is also small. Angle of prism ϕ is also small.

$$n=\frac{\sin(\psi+\phi/2)}{\sin\phi/2}\simeq\frac{(\psi+\phi)/2}{\phi/2}$$

Simplify and rearrange to obtain

$$n\phi=\psi+\phi$$

or $$\psi=(n-1)\phi$$

Thus, the angle of deviation is independent of angle of incidence.

41.10. $\phi=60$

$n=1.6$

(*a*) $$n=\frac{1}{\sin\alpha_2}$$

$$\sin\alpha_2=\frac{1}{1.6}=0.625$$

$$\alpha_2=39°$$

$$90-\alpha_1+90-\alpha_2=180-\phi$$

or $$\alpha_1=\phi-\alpha_2=60-39=21°$$

$$\frac{\sin\theta}{\sin\alpha}=n$$

$\therefore$ $\sin\theta=n\sin\alpha_1=1.6\times\sin 21°=1.6\times0.358=0.5728.$

or $$\theta=35°$$

(*b*) $\alpha_1+a_1=\phi$

Set $$\alpha_1=\alpha_2$$

$$2\alpha_1=\phi=60°$$

or $$\alpha_1=30°$$

$$\frac{\sin\theta}{\sin\alpha_1}=n$$

$\therefore$ $\sin\theta=n\sin\alpha_i=1.6\times\sin 30°=1.6\times0.5=0.8$

or $$\theta=54°.$$

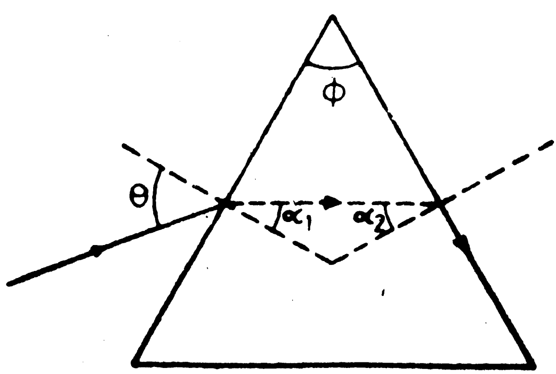


Fig. 41.10

41.11. $\theta_1=45°$

$$\sin\alpha_1=\frac{\sin\theta_1}{n}$$

$$\alpha_1=60°-\alpha_1$$

$\sin\theta_2=n\sin\alpha_2$; $\psi=\theta_1-\alpha_1+\theta_2-\alpha_2$

Light	λ	n	α_1	α_2	θ_2	ψ
Blue	4750 *A*	1.463	28°54′	31°06′	49°05′	34°05′
Yellow-green	5500 *A*	1.460	28°58′	31°02′	48°49′	33°49′
Red	6700 *A*	1.456	29°05′	30°57′	48°22′	33°29′

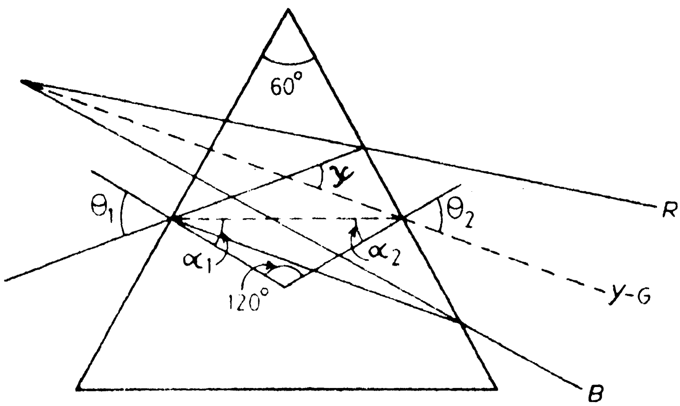


Fig. 41.11

41.12.

θ_1	$\sin\theta_1$	θ_2	$\sin\theta_2$	$n=\frac{\sin\theta_1}{\sin\theta_2}$	$\delta=\bar{n}-m$
10°	0.17365	7°45′	0.13437	1.292	+0.018
20°	0.34202	15°30′	0.26724	1.280	+0.030
30°	0.50000	22°30′	0.38268	1.307	+0.003
40°	0.64279	29°0′	0.48481	1.326	−0.016
50°	0.76604	35°0′	0.57358	1.336	−0.026
60°	0.86603	40°30′	0.64945	1.333	−0.023
70°	0.93969	45°30′	0.71325	1.317	−0.007
80°	0.98481	50°0′	0.76604	1.286	+0.024

$n=1.310 \pm 0.02$

δ=deviation from mean value

$\sigma=\sqrt{\overline{\sigma^2}}=0.020$

Observations are consistent with Snell's law.

41.13. See Fig 41-13

$\theta/\alpha=n$

or $\quad \alpha=\dfrac{\theta}{n}\ ;\ \sin(90-\theta)=\dfrac{x}{a}$ ...(1)

$\dfrac{b}{t}=\tan\alpha\simeq\alpha=\dfrac{\theta}{n}$ ...(2)

$\dfrac{(a+b)}{t}=\tan\theta=\theta$ ...(3)

$\cos\theta\simeq 1=\dfrac{x}{a}$ or $a=x$

$\therefore\quad a+b=x+\dfrac{\theta t}{n}=\theta t$

whence $\quad x=\theta t\,\dfrac{(n-1)}{n}$

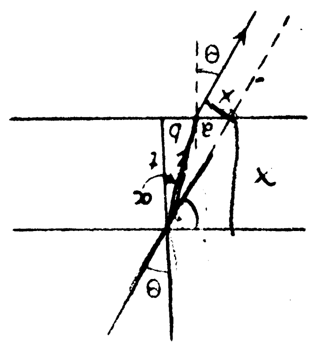


Fig. 41.14

41.14. For λ_{blue}, $n_B=1.463$

For λ_{red}, $\lambda n_R=1.455$

For blue light $\theta_{critical}=43.1°$.

For Red light $\theta_{critical}=43.4°$

(*a*) At $\theta\simeq43°$, blue light will be internally reflected but red light transmitted, as the critical angle for the red light is slightly greater. Therefore, when white light travels fused quartz, at around $\theta\simeq43°$' internal reflected light will contain blue component and red component will be transmitted.

(*b*) As the angle of incidence is allowed to increase the internally reflected light will contain both blue as well as red components. Separation of blue colour is not possible.

41.15. $\sin\theta_c=1/n$

The fraction of light energy that can escape is equal to the fraction of the solid angle through which the light can pass upward, towards the surface without total internal reflection. Let the light go through a cone whose half angle is equal to the critical angle θ_c such that $\sin\theta_c=1/n$, where n is the refractive index (Fig. 41 15).

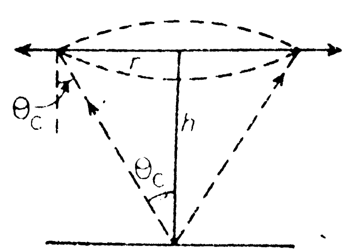


Fig 41.15

The fraction of solid angle

$$f=\frac{2\pi(1-\cos\theta_c)}{4\pi}\doteq\frac{1-\cos\theta_c}{2}$$

or $\quad f=\frac{1}{2}-\frac{1}{2}\sqrt{1-\sin^2\theta_c}=\frac{1}{2}-\frac{1}{2}n\sqrt{n^2-1}$

For $\quad n=1.3\,3,$

$$f=\frac{1}{2}-\frac{1}{2\times1.33}\sqrt{(1.33)^2-1}=0.17.$$

41.16. In Fig. 41.16 $\sin\theta_1=\frac{1}{2}n$

But $\frac{\sin\theta_1}{\sin\alpha}=n$

$\therefore\quad \sin\alpha=\frac{1}{2}$

or $\quad \alpha=30°\ ;\ \beta=60°$

$\therefore\quad \gamma=180-(75+60)=45°$

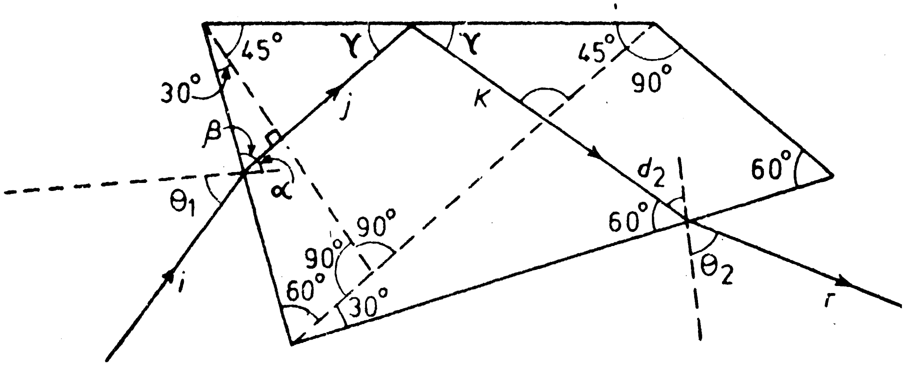


Fig. 41.16

$\delta=90°$

$\alpha_2=30°$

$$n=\frac{\sin\theta_2}{\sin\alpha_2}$$

$\therefore\quad \sin\theta_2=n/2=\sin\theta_1$

$\therefore\quad \theta_2=\theta_1$

It is sufficient to show that the normals at the surface on which the ray is incident and the one from which r emerges are at right angles. But this is so, as these normals are parallel to the two faces which are inclined at 90°.

41.17. Let n_g=refractive index of glsss in air.

n_l=refractive index of liquid in air.

n_{lg}=refractive index of liquid with respect to glass

$$n_{lg}=n_g/n_l \qquad ...(1)$$

At small angle θ incident ray will be able to penetrate liquid and escape into air. By increasing θ continuously, we can find when the total internal reflection occurs at air-liquid interface. Let the critical angle be θ_l. At the critical angle the ray suffers total internal reflection at the liquid-air surface and goes into the glass and can be observed.

$$n_l=\frac{1}{\sin\theta_l} \qquad ...(2)$$

If θ is allowed to increase then at a higher angle θ_{gl} the ray suffers internal reflection at the glass-liquid surface.

$$n_{gl}=\frac{1}{\sin\theta_{gl}} \qquad ...(3)$$

The refractive index of liquid is found out from (2). With the use of (1) and (3) together with the knowledge of n_l obtained from (2), the value of refractive index of glass (n_g) can be found out.

The method works provided $n_g > n_l$.

41.18. $$\frac{\sin 45°}{\sin\alpha}=n \qquad ...(1)$$

Also, $$n=\frac{1}{\sin\beta}=\frac{1}{\sin(90-\alpha)}=\frac{1}{\cos\alpha} \qquad ...(2)$$

From (1), $\sin\alpha=\frac{1}{\sqrt{2}\,n}$

$$\therefore\ \cos^2\alpha=1-\sin^2\alpha=1-\frac{1}{2n^2}=\frac{2n^2-1}{2n^2} \qquad ...(3)$$

Eliminate $\cos\alpha$ between (2) and (3)

$$\frac{1}{n^2}=\frac{2n^2-1}{2n^2}$$

i e. $2n^2-2n-1=0$

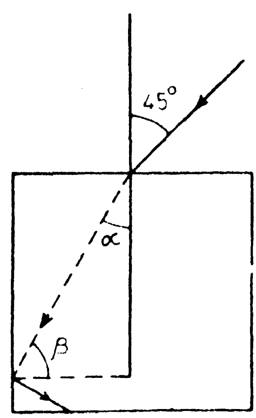


Fig. 41.18

$$n=\frac{2\pm\sqrt{4+8}}{4}=\frac{2\pm2\sqrt{3}}{4}=\frac{1}{2}(1+\sqrt{3})$$

$$n=1.366$$

41.19. $\sin\theta_c=\frac{1}{n}=\frac{1}{1.5}=0.6667$

$\therefore \qquad \theta_c = 41.8°$

$$\tan \theta_c = \frac{r}{0.5}$$

$r = 0.5 \tan \theta_c = 0.5 \tan 41.8°$
$= 0.5 \times 0.8941 = 0.447$ cm.

Area of each circle $= \pi r^2 = \pi(0.447)^2$
$= 0.627$ cm^2.

Area of each side $= 1.0$ cm^2.

The centre of each face must be covered with a circle of radius 0.45 cm.

Fraction of area covered is 0.63.

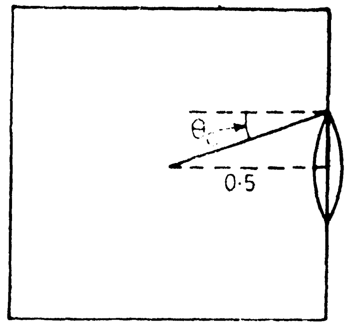


Fig. 41.19

41.20. Consider a ray to proceed from the point A and upon reflection at B in the plane P to arrive at the point C. (Fig. 41.20). Construct the normal N through A perpendicular to the plane P. Extend the normal N as far behind the mirror as the point A is in front of the mirror i.e. $A'D = AD$. Join AB and draw the straight line $A'BC$. As the triangles ABD and $A'BD$ are congruent, $AB = A'B$. Hence, the path $ABC = A'BC$. Now ABC is one possible path fron A to C via the reflecting surface. For any other path $AB'C$ is equal to $A'B'C$ which is greater than $A'BC$ since the side of a triangle is shorter than the sum of the two other sides. In accordance with Fermat's principle light follows the path ABC which is the shortest. It is obvious that any path from A to C via a point B'' on the plane P but outside the plane of paper will be greater than the straight line $A'BC$. Hence, the reflected ray lies in the plane of incidence, this being the plane containing the incident ray AB and the normal to the reflecting plane at B.

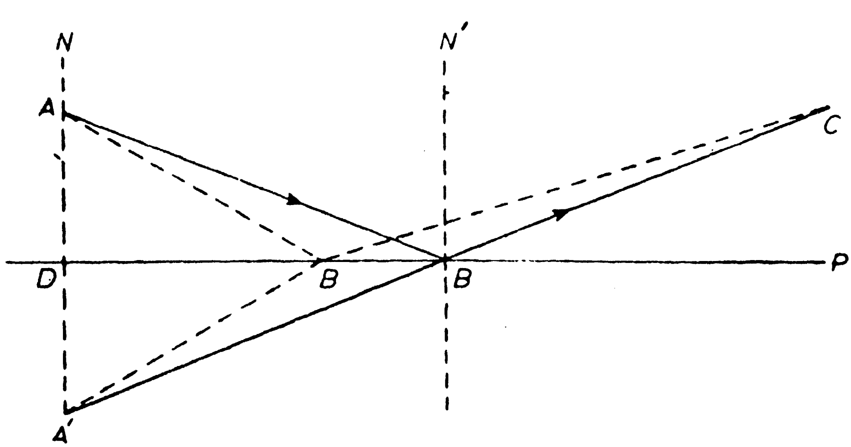


Fig. 41,20

41.21. The total optical path is

$$l=\sqrt{a^2+x^2}+\sqrt{b^2+(d-x)^2} \qquad \ldots(1)$$

$$\frac{dl}{dx}=\frac{1}{2}\times 2x\times(a^2+x^2)^{-1/2}+\frac{2}{2}(-1)(d-x)[b^2+(d-x)^2]^{-1/2}=0$$

$$\therefore \quad \frac{x}{\sqrt{a^2+x^2}}=\frac{d-x}{\sqrt{b^2+(d-x)^2}}$$

Solve for x to find $x=\dfrac{ad}{a+b}$ $\qquad \ldots(2)$

Use (2) in (1) to find

$$l=\sqrt{d^2+(a+b)^2} \qquad \ldots(3)$$

$$\frac{d^2l}{dx^2}=\frac{d}{dx}\frac{x}{\sqrt{x^2+a^2}}+\frac{d}{dx}\frac{(x-d)}{\sqrt{(x-d)^2+b^2}}$$

$$=\frac{\sqrt{x^2+a^2}\times 1-x\times\dfrac{1}{2}\dfrac{2x}{\sqrt{x^2+a^2}}}{x^2+a^2}+\frac{\sqrt{(x-d)^2+b^2}\times 1-(x-d)\times\dfrac{1}{2}\times\dfrac{2\times(x-d)}{\sqrt{(x-d)^2+b^2}}}{(x-d)^2+b^2}$$

$$=\frac{a^2}{(x^2+a^2)^{3/2}}+\frac{b^2}{[(x-d)^2+b^2]^{3/2}}$$

$$=\frac{(a+b)^4}{ab[d^2+(a+b)^2]^{3/2}}=\text{positive}$$

where we have used (2).

$\therefore$ l given by (3) is a minima.

Next consider the problem of refraction.

$$l=n_1l_1+n_2l_2=n_1\sqrt{a^2+x^2}+n_2\sqrt{b^2+(d-x)^2} \qquad \ldots(4)$$

$$\frac{dl}{dx}=n_1\times\frac{1}{2}\frac{2x}{\sqrt{a^2+x^2}}+\frac{n_2\times\frac{1}{2}\times 2(-1)(d-x)}{\sqrt{b^2+(d-x)^2}}$$

or $$\frac{dl}{dx}=\frac{n_1x}{\sqrt{a^2+x^2}}+\frac{n_2(x-d)}{\sqrt{b^2+(x-d)^2}} \qquad \ldots(5)$$

$$\frac{d^2l}{dx^2}=\frac{n_1\left[\sqrt{a^2+x^2}\times 1-x\times\dfrac{1}{2}\times 2x\times\dfrac{1}{\sqrt{a^2+x^2}}\right]}{(a^2+x^2)}$$

$$+n_2\frac{\left[\sqrt{b^2+(x-d)^2}\times 1-(x-d)\times\frac{1}{2}\times\frac{2\times(x-d)}{\sqrt{b^2+(x^2-d)^2}}\right]}{[b^2+(d-x)^2]}$$

$$\frac{d^2l}{dx^2}=\frac{n_1a^2}{(a^2+x^2)^{3/2}}+\frac{n_2b^2}{[b^2+(d-x)^2]^{3/2}} \qquad ...(6)$$

For any value of x, expression (6) is positive showing thereby t hat it is a minimum.

41.22. Optical path $l=AC+CB.n$

Now, $(AC)^2=(CD)^2+(AD)^2-2\,(AD)(CD)\cos\phi$

$$=R^2+(a+R)^2-2\,(a+R)\,R\,(1-\phi^2/2)$$

where we have expanded $\cos\phi$ upto two terms. Upon simplifying the right hand side we find,

$$(AC)^2=a^2+R(a+R)\,\phi^2$$

Similarly,

$$(BC)^2=a^2-R(a-R)\,\phi^2$$

$$\therefore \qquad l=\sqrt{a^2+R(a+R)\,\phi^2}+n\sqrt{a^2-R(a-R)\,\phi^2}$$

$$\frac{\partial l}{\partial\phi}=\frac{1}{2}\times\frac{2\,\phi\,R(a+R)}{\sqrt{a^2+R(a+R)\phi}}+\frac{n\times\frac{1}{2}\times 2\,\phi\,R(R-a)}{\sqrt{a^2+R(R-a)\phi^2}} \qquad ...(1)$$

Setting $\partial l/\partial\phi=0$ as the condition for maximum or minimum, we find

$$\frac{a+R}{\sqrt{a^2+R(a+R)\phi^2}}=\frac{n(a-R)}{\sqrt{a^2+R(R-a)\phi^2}} \qquad ...(2)$$

Differentiate (1) with respect to ϕ,

$$\frac{\partial^2 l}{\partial\phi^2}=\frac{R(a+R)\left[\sqrt{a^2+R(a+R)\phi^2}-\frac{\phi.2\phi R\,(R-a)}{\sqrt{a^2+R(a+R)\phi^2}}\right]}{[a^2+R(a+R)\phi^2]}$$

$$+nR(R-a)\frac{\left[\sqrt{a^2+R(R-a)\phi^2}-\frac{\phi.2\phi\times\frac{1}{2}\,R(R-a)}{\sqrt{a^2+R(R-a)^2\phi}}\right]}{[a^2+R(R-a)\phi^2]}$$

$$\text{or,} \qquad \frac{\partial^2 l}{\delta\phi^2}=\frac{a^2R(a+R)}{[a^2+R(a+R)\phi^2]^{3/2}}+\frac{a^2R(R-a)\,n}{[a^2+R(R-a)\phi^2]^{3/2}} \qquad ...(3)$$

Use (2) to eliminate the dinominator of the second term in (3). Then (3) becomes

$$\frac{\delta^2 l}{\delta\phi^2}=\frac{a^2R(a+R)}{[a^2+R(a+R)\phi^2]^{3/2}}\left[1-\frac{(a+R)^2}{n^2(a-R)^2}\right]$$

As the expression outside the square bracket is always positive, the sign of $\frac{\delta^2 l}{\delta\phi^2}$ depends entirely on the sign of the expression in the square bracket. Thus, $\frac{\delta^2 l}{\delta\phi^2}$ is +ve (condition for minimum)

if $$\frac{(a+R)^2}{n^2(a-R)^2} < 1$$

or $$\frac{a+R}{n(a-R)} < 1$$

i.e. $a+R < n\,(a-R)$

or $$a(1-n) < -R(n+1)$$

or $$a < R\frac{(n+1)}{(n-1)}$$

Similarly, condition for l to be stationary (not changing) is

$$a = \frac{R(n+1)}{(n-1)}$$

and for l to be maxima,

$$a > R\frac{(n+1)}{(n-1)}.$$

SUPPLEMENTARY PROBLEMS

S.41.1. The opening angle θ for the Cerenkov radiation is given by

$$\cos\theta = \frac{1}{\beta n} \qquad ...(1)$$

with $\beta = v/c$ and n the index of refraction.

Minimum speed of electron is obtained by setting $\theta=0$ in (1), in which case

$$\beta_{(min)} = \frac{1}{1.54} = 0.649$$

whence $v_{(min)} = 0.649\ c = (0.649)(3\times10^8$ meter/sec)

$= 1.95\times10^8$ meter/sec.

S.41.2. (*a*) Imagine that the atmosphere is divided into horizontal layers of increasing index of refraction in going from top to bottom as in Fig. S.41.2. Let the rays of light from a star be incident at an angle θ_1 with respect to the zenith. As the ray traverses various layers of atmosphere bending occurs in accordance with Snell's law. The curved path which the ray travels is approximated as a series of straight lines in various layers. The bending takes place towards the normal in each layer as the index of refraction progressively increases in traversing down. Applying Snell's law to various layers,

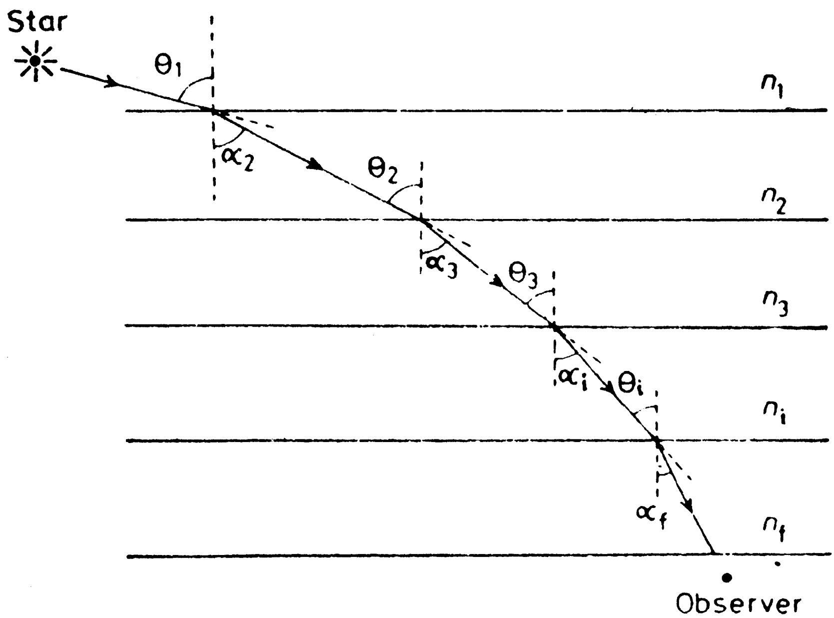


Fig. S.41.2

$n_1 \sin \theta_1 = n_2 \sin \alpha_2$ ...(1)

$n_2 \sin \theta_2 = n_3 \sin \alpha_3$ ...(2)

....

....

$n_i \sin \theta_i = n_f \sin \alpha_f$ ...(i)

Add equations (1), (2)...(i). Note that $\theta_2 = \alpha_2$, $\theta_3 = \alpha_3 \cdots$ we find,

$$n_1 \sin \theta_1 = n_f \sin \alpha_f$$

where α_f is the apparent angle of the star and n_f the index of refraction at the point of observation. Furthermore, we can set $n_1 = 1$ as the uppermost layer of the atmosphere is so tenuous that it is as good as vacuum.

We therefore find,

$$\sin \alpha_f = \frac{\sin \theta_1}{n_f}$$

showing thereby that the apparent angle of a star is independent of how n varies with altitude and depends only on its value at the earth's surface and on the incident direction.

(*b*) Owing to the curvature of earth, the atmospheric layers of uniform refractive index are no longer horizontal slices but are now spherical shells. The analysis is complicated by the fact that the normals at various layers are no longer parallel but tend to converge at the center of the earth. The angle of incidence at one layer

of atmosphere is no longer equal to the angle of refraction at the previous layer. Thus, unlike the previous analysis, cancellation of various terms is no longer possible. In particular, the angle of the star with the zenith will be larger and will not be independent of the variation of refractive index with altitude.

S.41.3. The critical angle is given by

$$\theta_c = \sin^{-1}\frac{1}{n} = \sin^{-1}\frac{1}{1.33}$$

$$= 49°.$$

The radius of the circle is

$$r = (80 \text{ cm}) \tan\theta_c$$

$$= (80 \text{ cm})(\tan 49°) = 92 \text{ cm}.$$

Therefore, diameter of the largest circle is $2r = 184$ cm.

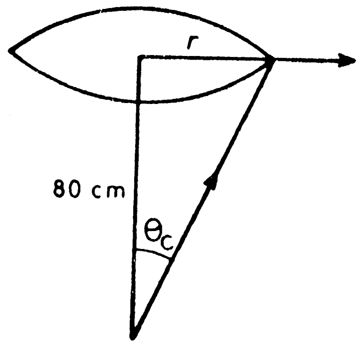


Fig. S.41.3

S.41.4. From Fig S.41.4,

$$\frac{\sin\theta}{\sin r} = n$$

$$\therefore \quad \sin r = \frac{\sin\theta}{n} = \frac{\sin 45°}{1.33} = 0.5316$$

or $\quad \hat{r} = 32°.$

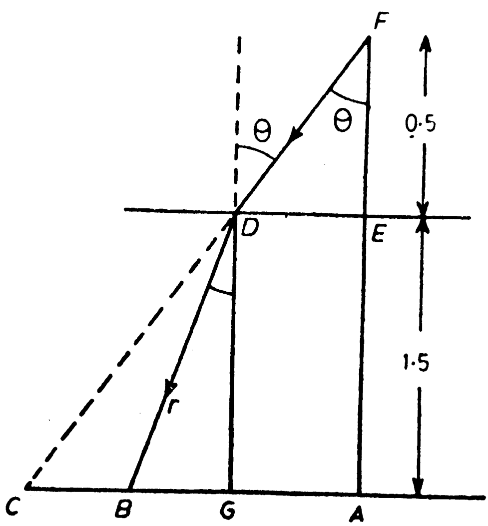


Fig. S.41.4

$BG = DG \tan r = (1.5 \text{ meter}) \tan 32° = 0.94$ meter

$GA = DE = EF = 0.5$ meter

$\therefore \quad AB = BG + GA = (0.94 + 0.5)$ meter $= 1.44$ meters.

Thus, the length of the shadow at the bottom of the pool is 1.44 meters.

S.41.5. (*a*) From Fig. S.41.5 (*a*),

$$\frac{\sin \theta_1}{\sin r} = n \qquad \ldots(1)$$

$$\sin \theta_c = \frac{1}{n} \qquad \ldots(2)$$

$$\theta_c = 90 - r \qquad \ldots(3)$$

$\therefore \ \sin \theta_c = \sin (90 - r) = \cos r$

$$\cos r = 1/n \qquad \ldots(4)$$

where use has been made of (2).

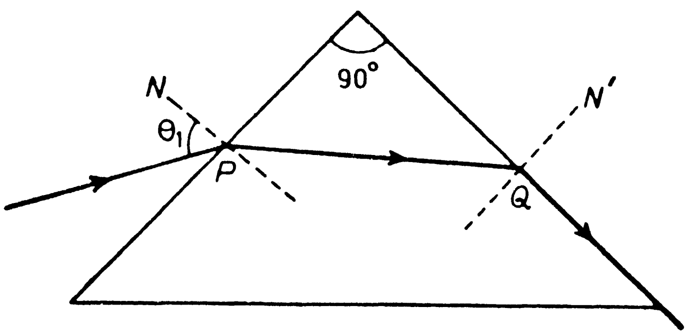


Fig. S.41.5 (a)

From (1), we have

$$\sin r = \frac{\sin \theta_1}{n} \qquad \ldots(5)$$

Squaring (4) and (5) and adding,

$$\cos^2 r + \sin^2 r = 1/n^2 \, (1 + \sin^2 \theta_1) = 1$$

Hence, $n = \sqrt{1 + \sin^2 \theta_1}$

(*b*) The maximum value of θ_1 can be 90°. $\theta_{1(max)}=90°$

$\sin\theta_{1(max)}=\sin 90°=1$

$n_{(max)}=\sqrt{1+1}=\sqrt{2}$

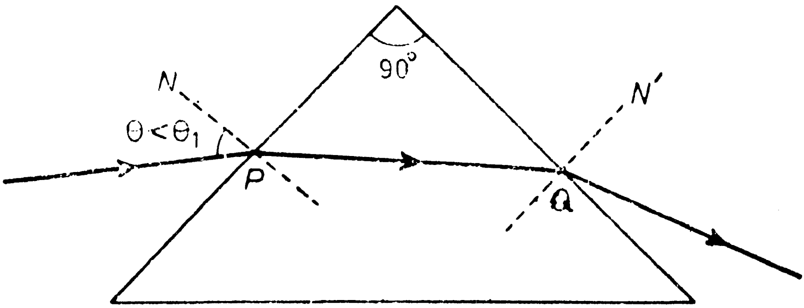


Fig. S.41.5 (b)

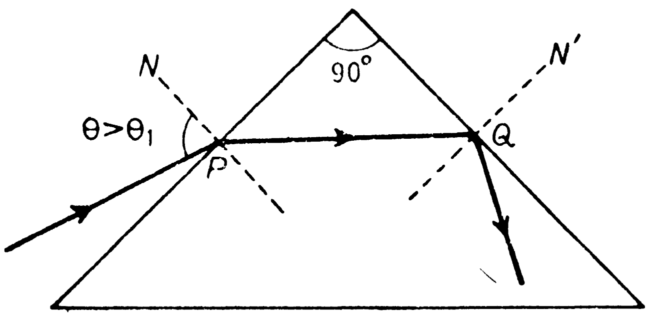


Fig. S.41.5. (c)

(*c*) For $\theta>\theta_1$, ray diagram is shown in Fig. S.41.5 (*b*). The ray emerges at the other side of the prism.

For $\theta<\theta_1$, the ray diagram is shown in Fig. S.41.5 (*c*). The ray suffers internal reflection.

S.41.6. (*a*) For normal incidence the ray passes undeviated through the air-water interface. From the geometary of Fig. S.41.6 (*a*), the angle of incidence as well as the angle of reflection at both the mirrors will be 45°.

Consequently, the angle of total deviation will be 180°. In other words the emerging ray will be antiparallel to the incident ray and will go out undeviated as it falls normal on the water-air interface.

(*b*) Let the angle of incidence at the first and second mirror be θ and α respectively.

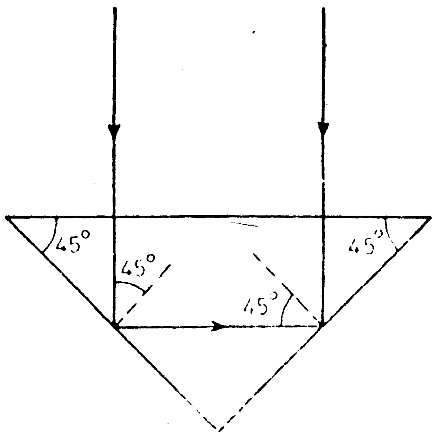


Fig. S.41.6 (a)

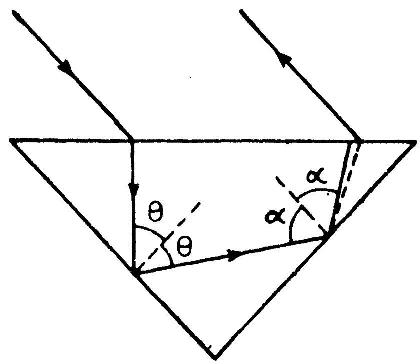


Fig. S.41.6. (b)

From the geometry of Fig S.41.6 (*b*) it is obvious that $\theta+\alpha=90°$.

Total deviation, $\delta=\pi-2\theta+\pi-2\alpha$

$$=2\pi-2\,(\theta+\alpha)$$

$$=2\pi-2\left(\frac{\pi}{2}\right)=\pi$$

Thus, in water the incident ray and the second reflected ray must be antiparallel. Hence the ray incident obliquely on air-water interface and the emergent ray must also be antiparallel.

(*c*) The three dimensional analog to the above problem is the arrangement of three plane mirrors hinged at right angle to earth other like two adjacent walls and the ceiling of a room. A light ray incident on any of the mirrors after single or multiple reflections will emerge as parallel to the incident ray. The proof is an extension of that given for two mirror problem.

42 REFLECTION AND REFRACTION—SPHERICAL WAVES AND SPHERICAL SURFACES

42.1. (*a*) $\theta=45°$

$$n=\frac{2\pi}{\theta}-1=\frac{360}{45}-1=7$$

(*b*) $\theta=60°$

$$n=\frac{360}{60}-1=5$$

(*c*) $\theta=120°$

$$n=\frac{360}{120}-1=2$$

42.2. $\theta=90°$

$$n=\frac{360}{90}-1=3$$

42.3. Since the image will be 10 cm behind the mirror, the distance between the observer (who is standing 30 cm in front of the mirror) and the image of the object, will be 30+10=40 cm. Hence, the observer must focus his eyes at a distance of 40 cm.

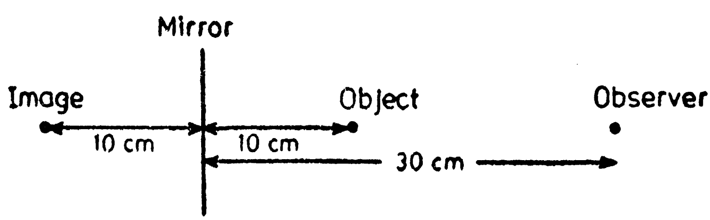


Fig. 42.3.

42.4. O_1 and O_2 are the images of object O respectively by mirrors M_1 and M_2. Rays are shown for this image formation O_3 and O_4 are images of images respectively of O_1 by mirror M_2 and O_2 by mirror M_1 (rays not shown for the simplicity of the figure). Obviously, $OM_1=O_1M_1$, $OM_2=O_2M_2$, $O_1M_2=O_3M_2$ $O_2M_1=O_4M_1$, with $2\,OM_1=OM_2$.

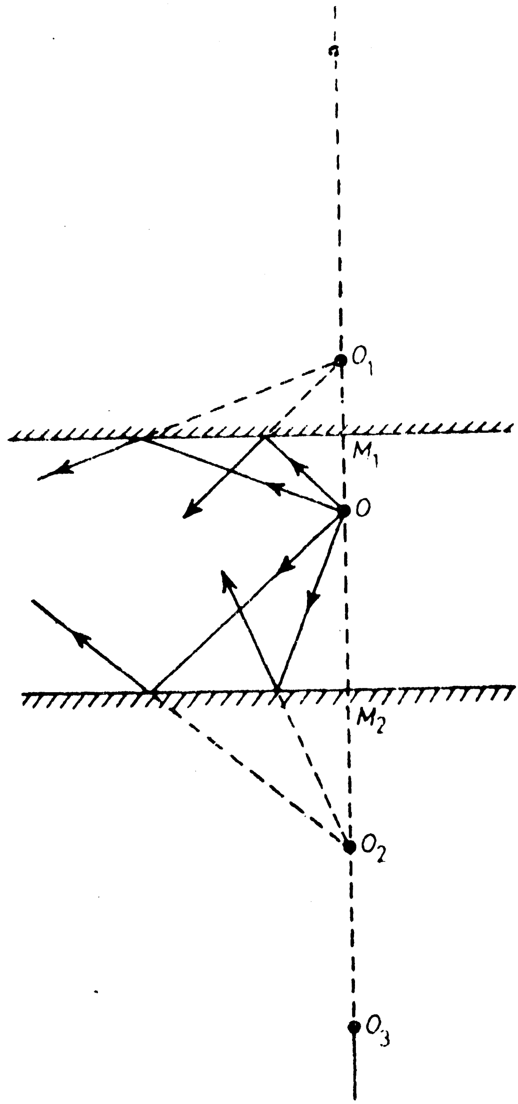


Fig. 42.4

42.5. Paraxial rays, i.e. bundle of rays close to the axis upon striking the spherical mirror are brought to a sharp focus at F in the focal plane. However, rays which travel further from the axis do not form the image at a common point, give rise to spherical aberration. In Fig. 42.5 (a) are shown parallel incident rays at large distance h from the axis which upon reflection cross the axis closer to the mirror. Along the axis the size of the circular image is of least size, This is called circle of least confusion,

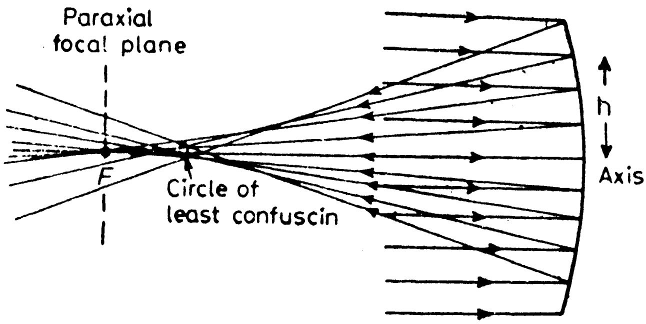


Fig. 42.5 (a)

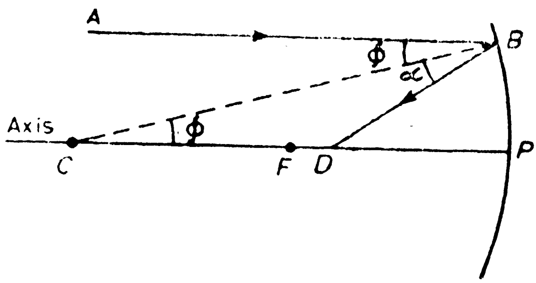


Fig. 42.5 (b)

That the parallel rays away from the axis upon reflection from the mirrors cross the axis inside the paraaxial focal point F can be easily proved with reference to Fig. 42.5 (b). According to the law of reflection the incident ray AB after reflection goes along BD such that angle of reflection α is euqal to ϕ, the angle of incidence which in turn is equal to $B\hat{C}P$. It follows that triangle BCD is isosceles so that $CD=DB$. Now in a triangle a side is smaller than the sum of the other two sides. In triangle BCD,

$$BC<CD+DB$$

Now BC is the radius of the mirror and is equal to CP.

Therefore, $CP < 2CD$

or $\frac{1}{2}CP < CD$

But $\frac{1}{2}CP=CF$, since focal length is half of the radius of curvature.

Thus, $CF<CD$

Therefore, the point D lies within the paraaxial focus F. As the point B moves toward P, the point D approaches F.

42.6. Following formulas are useful for completing the information in the Table regarding the spherical mirrors.

$$\frac{1}{o}+\frac{1}{i}=\frac{1}{f}$$

$$m=-\frac{i}{o}$$

$$r=2f$$

	a	b	c	d	e	f	g	h
Type	*concave*	*plane*	*concave*	*concave*	*convex*	*convex*	*convex*	*concave*
f	+20	∞	+20	+20	−20	−20	−20	+ 8
r	+40	∞	+40	+40	−40	−40	−40	+16
i	−20	−10	+60	+30	−10	−18	−4	+12
o	+10	+10	+30	+60	+20	−180	+5	+24
m	+ 2	+ 1	+ 2	−0.5	+0.5	+0.1	+0.8	−0.5
Real Image	No	No	Yes	Yes	No	No	No	Yes
Erect Image	Yes	Yes	No	No	Yes	Yes	Yes	No

The results arc in agreement with the graph shown in Textbook Fig. 42.16.

42.7. $\frac{1}{o}+\frac{1}{i}=\frac{1}{f}$ ...(1)

Longitudinal magnification (m')

(*a*) $m'=\frac{di}{d_0}=-\frac{i^2}{o^2}=-m^2$

where $m=\frac{i}{o}$=transverse or lateral magnification.

$$m'=\frac{l'}{l}=-m^2$$

Multiply by (1) through by o and re-arrange to find

$$\frac{i}{o}=m=\frac{f}{o-f}$$

$$\therefore \quad m'=\frac{l'}{l}=\left[\frac{f}{o-f}\right]^2$$

Ignoring the sign

$$l'=l\left[\frac{f}{o-f}\right]^2$$

(*b*) Differentiate (1) holding f=constant.

$$-\frac{do}{o^2}-\frac{di}{i^2}=0$$

or $$\frac{di}{do}=-\frac{i^2}{o^2}=-m^2$$

$\therefore$ $$m'=\frac{l'}{l}=m^2, \text{ ignoring the sign.}$$

(*c*) Since transverse magnification m=longitudinal magnification, m'

$$m'=m$$

But $$m'=m^2$$

$\therefore$ $$m^2=m$$

or $$m=1$$

$\therefore$ $$m=\frac{f}{o-f}=1 \text{ or } o=2f=R$$

Therefore, object must be placed at the centre of the curvature of the mirror.

42.8 (*a*) When viewed normally,

$$n=\frac{\text{actual depth}}{\text{apparant depth}}$$

$\therefore$ apparent depth

$$=\frac{\text{actual depth}}{n}$$

$$=\frac{8.0 \text{ ft}}{1.33}=6.0 \text{ ft.}$$

(*b*) $$\frac{\sin 30^\circ}{\sin r}=1.33$$

whence $$\sin r=\frac{0.5}{1.33}=0.376$$

or $$r=22^\circ$$

$$x=8.0\times\tan r$$
$$=8.0\times\tan 22^\circ$$
$$=8.0\times0.4=3.2 \text{ ft.}$$

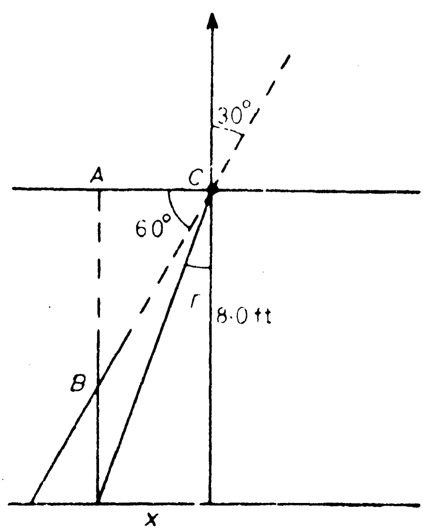


Fig. 42.8.

$$AB=AC\tan 60^\circ=3.2\times1.73=5.54 \text{ ft.}$$

42.9. $$\frac{\sin\theta}{\sin r}=n_1=1.33$$

$$\tan\theta=\frac{CD}{BC}$$

$$\tan r_1 = \frac{FH}{DH} = \frac{FH}{2.0}$$

$$\tan r_2 = \frac{AG}{FG} = \frac{AG}{4.0}$$

$$AG + FH = CD$$

$$\frac{\sin r_1}{\sin r_2} = \frac{n_2}{n_1} = \frac{1.46}{1.33} = 1.1$$

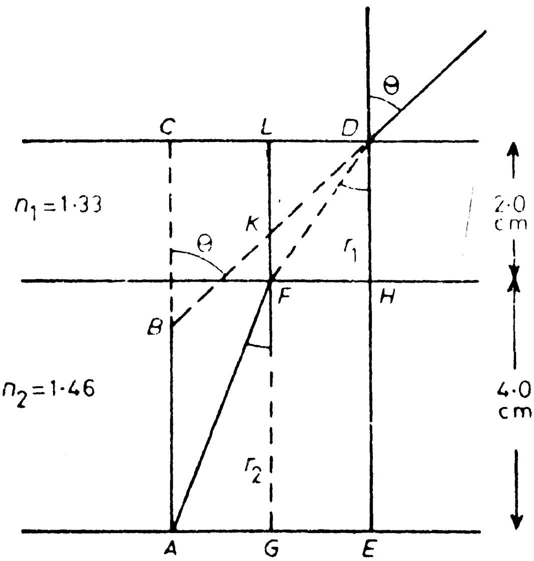


Fig. 42.9

Consider angles θ, r_1, and r_2 to be small so that $\tan\theta = \theta$ etc.,

$$\theta = 1.33\ r_1$$

$$r_1 = \frac{FH}{2.0}$$

$$FH = 2.0\ r_1$$

$$r_2 = \frac{AG}{4.0}$$

$$r_1/r_2 = 1.1$$

$$AG = 4r_2 = 4r_1/1.1$$

$$BC = \frac{CD}{\theta} = \frac{FH+AG}{\theta} = \frac{FH+AG}{1.33\ r_1}$$

$$= \frac{1}{1.33}\left(2.0 + \frac{4}{1.1}\right) = 4.23\ \text{cm.}$$

42.10. A useful formula for filling up the given table with information concerned with a spherical surface separating two media with different indices of refraction is

$$\frac{n_1}{o}+\frac{n_2}{i}=\frac{n_2-n_1}{r}$$

	a	b	c	d	e	f	g	h
n_1	1.0	1.0	1.0	1.0	1.5	1.5	1.5	1.5
n_2	1.5	1.5	1.5	Indeterminate	1.0	1.0	1.0	1.0
o	+10	+10	+71	+20	+10	+10	+ 70	+100
i	−12	−13	+600	−20	−6	−7.5	−105	+600
r	+30	−32.5	+ 30	−20	+30	−30	+ 30	− 3 0
Real Image	No	No	Yes	No	No	No	No	Yes

The rays are shown in Fig 42.10 for the situations (a), (b)......(h).

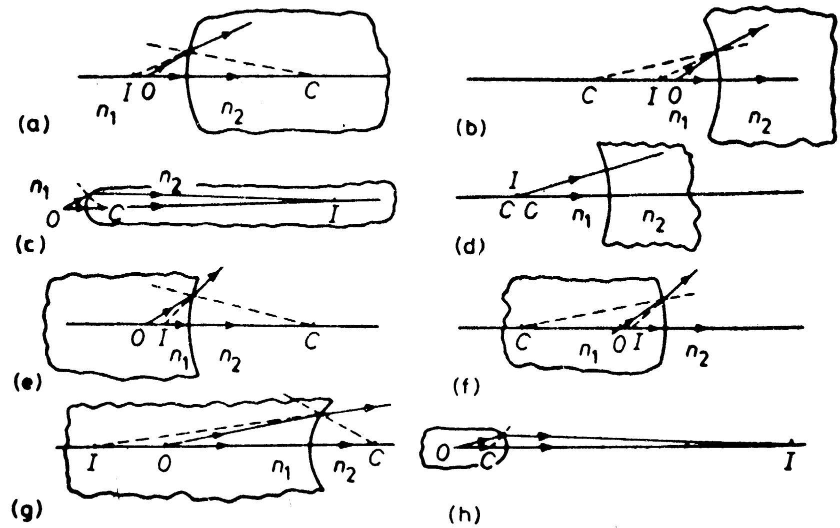


Fig. 42.10

42.11. Following formulas are useful in completing the information in the given table regarding thin lenses.

$$\frac{1}{o}+\frac{1}{i}=\frac{1}{f}$$

$$\frac{1}{f}=(n-1)\left(\frac{1}{r'}-\frac{1}{r''}\right)$$

$$m=-\frac{i}{o}$$

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*
Type	*converging*	*converging*	*converging*	*diverging*	*converging*	*diverging*	*diverging*	*diverging*	*converging*
f	+10	+10	+10	−10	+30	−30	−120	−10	+3.3
r'	—	—	—	—	−30	−30	−30	—	—
r''	—	—	—	—	+30	+30	−60	—	—
i	+20	−10	−10	−3.3	−15	−7.5	−9.2	−5	+5
o	+20	+5	+5	+5	+10	+10	+10	+10	+10
n	—	—	—	—	1.5	1.5	1.5	—	—
m	−1	+2	>1	<1	+1.5	+0.75	+0.92	+0.5	−0.5
Real image	yes	no	no	no	no	no	no	no	yes
Erect image	no	yes	yes	yes	yes	yes	yes	yes	no

The results are in agreement with the graph shown in Textbook Fig. 42.27.

42.12. $\frac{n_1}{o}+\frac{n_2}{i}=\frac{n_2-n_1}{r}$

Set $\quad n_1=1$ for air
$\quad n_1=n$

$$\frac{1}{o}+\frac{n}{i}=\frac{n-1}{r}$$

If $\quad o=\infty$, then $i=nr/(n-1)=f_2$.

This value of i gives the distance of the second principal focus from the pole. On the other hand if $i=\infty$ i.e. the beam becomes parallel after refraction, in that case $n/i=0$ and $o=r/(n-1)=f_1$. This value of o gives the distance of first principal focus from the pole and is called the first focal length of the surface.

42.13. (*a*) $\frac{1}{o_1}+\frac{1}{i_1}=\frac{1}{f}$ ...(1)

$$\frac{1}{o_2}+\frac{1}{i_2}=\frac{1}{f} \quad \text{...(2)}$$

$$i_1+o_1=i_2+o_2=D \quad \text{...(3)}$$

$$i_1-i_2=o_2-o_1=d \quad \text{...(4)}$$

From (3), $i_1=D-o_1$ ...(5)

$$i_2=D-o_2 \quad \text{...(6)}$$

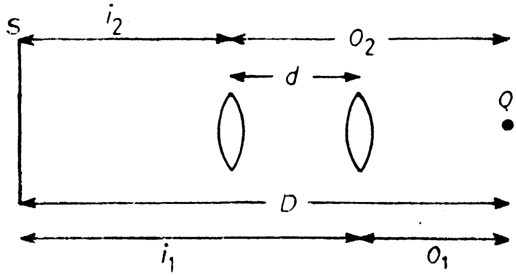


Fig. 42.13

Use (5) in (1) and (6) in (2),

$$\frac{1}{o_1}+\frac{1}{D-o_1}=\frac{1}{f} \quad \text{...(7)}$$

$$\frac{1}{o_2}+\frac{1}{D-o_2}=\frac{1}{f} \quad \text{...(8)}$$

Subtract (8) from (7),

$$\frac{1}{o_1}-\frac{1}{o_2}+\frac{1}{D-o_1}-\frac{1}{D-o_2}=0$$

which simplifies to

$$D[D-(o_1+o_2)]=0$$

$$D\neq 0$$

$\therefore$ $\quad D=o_1+o_2$ ...(9)

also, $\quad d=o_2-o_1$ ...(4)

Solve for o_1 and o_2,

$$o_1=\frac{D-d}{2} \qquad ...(10)$$

$$o_2=\frac{D+d}{2} \qquad ...(11)$$

Use (10) in (7) o obtain,

$$d=\sqrt{D(D-4f)}$$

(*b*) $\quad m_1=\frac{i_1}{o_1}=\frac{D-o_1}{o_1}$ ...(12)

$$m_2=\frac{i_2}{o_2}=\frac{D-o_2}{o_2} \qquad ...(13)$$

Use (10) in (12) and (11) in (13) to get

$$\frac{m_2}{m_1}=\left(\frac{D-d}{D+d}\right)^2$$

42.14. $\frac{1}{f}=(n-1)\left(\frac{1}{r'}-\frac{1}{r''}\right)$

(*a*) Here $r''=\infty$

$\therefore$ $\quad \frac{1}{f}=\frac{(n-1)}{r'}$

Centre of curvature C' lying on the R-side is +ve. So r' is positive.

$\therefore$ $\quad f=$ +ve. Hence, it is a converging lens.

(*b*) $\quad r'=\infty$

$\therefore$ $\quad \frac{1}{f}=-\frac{(n-1)}{r''}$

$r''=$ +ve, since C'' lies on R-side.

Hence, $f=$ −ve, that is, it is a diverging lens.

(*c*) Both r' and r'' are +ve as C' and C'' lie on the R-side. We note that $r' < r''$. Therefore, the quantity $\left(\frac{1}{r'}-\frac{1}{r''}\right)$ is +ve. Thus, $f=$ +ve. Hence, it is a converging lens.

(*d*) Both r' and r'' are +ve as C' and C'' lie on the R-side. But $r' > r''$.

$\therefore \qquad \left(\frac{1}{r'}-\frac{1}{r''}\right)$ is negative

$\therefore \qquad f=-$ve. Hence, it is diverging lens.

42.15. Let the object be placed at distance o from the first lens of focal length f_1. Let this lens produce an image at a distance i_1.

$$\frac{1}{o}+\frac{1}{i_1}=\frac{1}{f_1} \qquad \ldots(1)$$

The image at i_1 serves as a virtual object for the second lens of focal length f_2. Let a real image be formed at i_2, then,

$$-\frac{1}{i_1}+\frac{1}{i_2}=\frac{1}{f_2} \qquad \ldots(2)$$

Add (1) and (2) to obtain

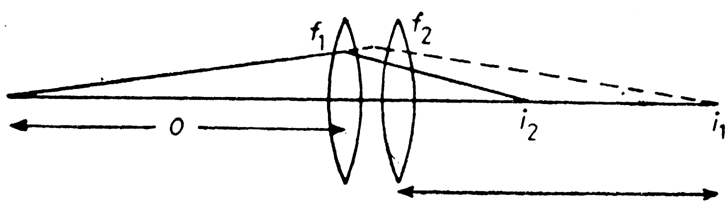


Fig. 42.15

$$\frac{1}{o}+\frac{1}{i_2}=\frac{1}{f_1}+\frac{1}{f_2} \qquad \ldots(3)$$

Now, consider the combination to be equivalent to one single lens of focal length F. Then,

$$\frac{1}{o}+\frac{1}{i_2}=\frac{1}{F} \qquad \ldots(4)$$

Combine (3) and (4) to find

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}=\frac{f_1+f_2}{f_1 f_2}$$

$$\therefore \qquad F=\frac{f_1 f_2}{f_1+f_2}$$

42.16. Let $D=o+i$

or $\qquad i=D-o \qquad \ldots(1)$

$$\frac{1}{o}+\frac{1}{i}=\frac{1}{f} \qquad \ldots(2)$$

Use (1) in (2) to eliminate i

$$\frac{1}{o}+\frac{1}{D-o}=\frac{1}{f} \qquad \ldots(3)$$

re-arranging the equation,

$$o^2 - oD + Df = 0$$

Solving the quadratic equation

$$o = \frac{D \pm \sqrt{D^2 - 4\,Df}}{2}$$

In order that o be real, the expression $(D^2 - 4\,Df)$ under the radical must be +ve, that is, $D^2 \geqslant 4\,Df$ or $D \geqslant 4f$.

42.17. $\dfrac{1}{f} = (n-1)\left(\dfrac{1}{r'} - \dfrac{1}{r''}\right)$

$n = 1.50$; $f = 6.0$ cm

$r'' = 2\,r'$; r' is +ve since it is on R-side.

r'' is −ve since C'' is on V-side.

$$\frac{1}{6} = (1.5 - 1)\left(\frac{1}{r'} + \frac{1}{2r'}\right)$$

Solve for r' ; we fiind $r' = 4.5$ cm.

$r'' = 9.0$ cm.

42.18. $\dfrac{1}{o} + \dfrac{1}{i} = \dfrac{1}{f}$ ...(1)

Set $x = o - f$ or $o = x + f$ ...(2)

$x' = i - f$ or $i = x' + f$ ...(3)

Substitute o and i from (2) and (3) in (1) to obtain,

$$\frac{1}{x+f} + \frac{1}{x'+f} = \frac{1}{f}$$

$$\frac{x + x' + 2f}{(x+f)(x'+f)} = \frac{1}{f}$$

Cross multiply and simplify to find the desired result,

$$f^2 = xx'$$

42.19. Let ϕ be the angle of incidence and θ the angle of refraction at A. Let r be the radius of the sphere and n the index of refraction of the material of the sphere. Obviously $A\hat{B}C = \theta$ and $E\hat{B}F = \phi$. Since DA is very close to GC, both θ and ϕ are small.

As $\sin\phi = n \sin\theta$, we can write

$$\phi = n\theta \quad ...(1)$$

Now $A\hat{C}G = \phi$

$$B\hat{C}F = \pi - B\hat{C}A - \phi$$

$$=\pi-(\pi-2\theta)-\phi$$
$$=2\theta-\phi$$

Also $B\hat{F}C=E\hat{B}F-B\hat{C}F$
$$=\phi-(2\theta-\phi)$$
$$=2(\phi-\theta)$$

In $\triangle$ *FBC*,

$$\frac{FC}{BC}=\frac{\sin FBC}{\sin BFC}=\frac{\sin\phi}{\sin 2(\phi-\theta)}=\frac{\phi}{2(\phi-\theta)}$$

Thus the equivalent focal length,

$$FC=BC\frac{\phi}{2(\phi-\theta)}=\frac{nr}{2(n-1)}$$

$$\therefore \quad FH=FC-HC=\frac{nr}{2(n-1)}-r=\frac{r(2-n)}{2(n-1)}$$

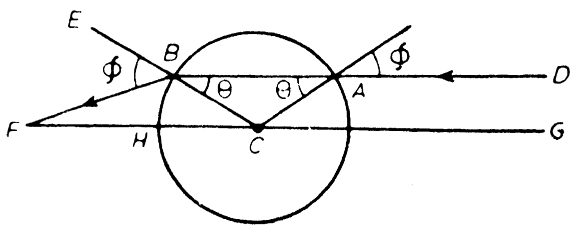


Fig. 42.19

42.20. $\frac{1}{o}+\frac{1}{i_1}=\frac{1}{f_1}$

$$\frac{1}{2f_1}+\frac{1}{i_1}=\frac{1}{f_1} \qquad \therefore\ i_1=2f_1$$

The first image is located at a distance $2f_2$ from the pole of concave mirror i.e. at the centre of curvature.

$$[2(f_1+f_2)-2f_1=2f_2]$$

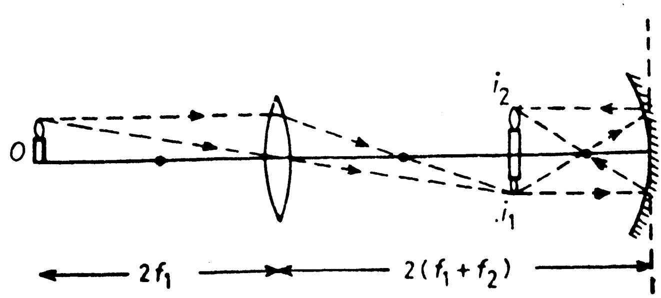


Fig. 42.20

The first image serves as an object for the concave mirror

$$\frac{1}{o_1}+\frac{1}{i_2}=\frac{1}{f_2}$$

$$\frac{1}{2f_2}+\frac{1}{i_2}=\frac{1}{f_2}$$

or $$i_2=2f_2$$

The image will be real, erect and at the same place as i_1 ; $m=1$.

42.21. (*a*) C_1 is on *V*-side and so r_1 is $-$ve.

C_2 is on *R*-side and so r_2 is $+$ve.

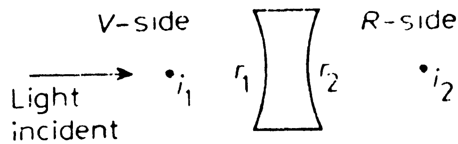


Fig. 42.21 (a)

(*b*) $$\frac{1}{f}=\frac{1}{o}+\frac{1}{i}=(n-1)\left(-\frac{1}{r}-\frac{1}{r}\right)$$

or $$\frac{1}{r}+\frac{1}{i}=-\frac{2}{r}(n-1)$$

Solve to find, $$i=\frac{r}{1-2n}$$

(*c*) The image is virtual and erect.

(*d*) The ray diagram is shown below in Fig. 42.21 (*b*).

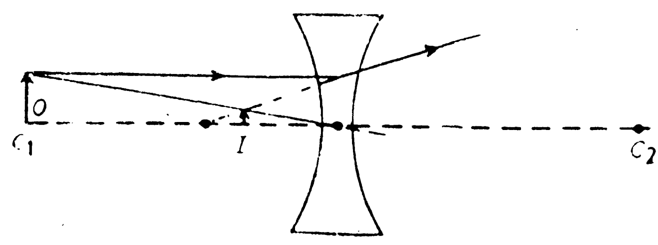


Fig. 42.21 (b)

42.22. From the Textbook Fig. 42.32 we find

$f_1=7$ mm

$f_2=25$ mm

$o_1=10$ mm (object distance fron lens 1)

$d=46$ mm (distance of separation of lenses)

$$\frac{1}{o_1}+\frac{1}{i_1}=\frac{1}{f_1}$$

or $$i_1=\frac{o_1 f_1}{o_1-f_1}=\frac{(10\text{ mm})(7\text{ mm})}{(10\text{ mm}-7\text{mm})}=23.33\text{ mm}.$$

The first image due to lens 1 is formed 23.33 mm behind the lens 1 i.e. $o_2=(46\text{ mm}-23.33\text{ mm})$ or 22.67 mm in front of lens 2. The image is real, magnified and inverted. The magnification is given by $m_1=i_1/o_1=23.3\text{ mm}/10\text{ mm}=2.3$. This image acts as an object for lens 2. The image distance for the second lens is calculated from

$$\frac{1}{o_2}+\frac{1}{i_2}=\frac{1}{f_2}$$

whence $i_2=-\dfrac{(25\text{ mm})(22.67\text{ mm})}{(25\text{ mm}-22.67\text{ mm})}=-243\text{ mm or }-24.3\text{ cm}.$

in front of lens 2. The negative sign shows that the image is virtual. As the image i_2 remains upright with respect to i_1, the final image i_2 is inverted as i_1 is already inverted. The magnification due to lens 2 alone is

$$m_2=i_2/o_2=247\text{ mm}/22.7\text{ mm}=10.9$$

The overall magnification is

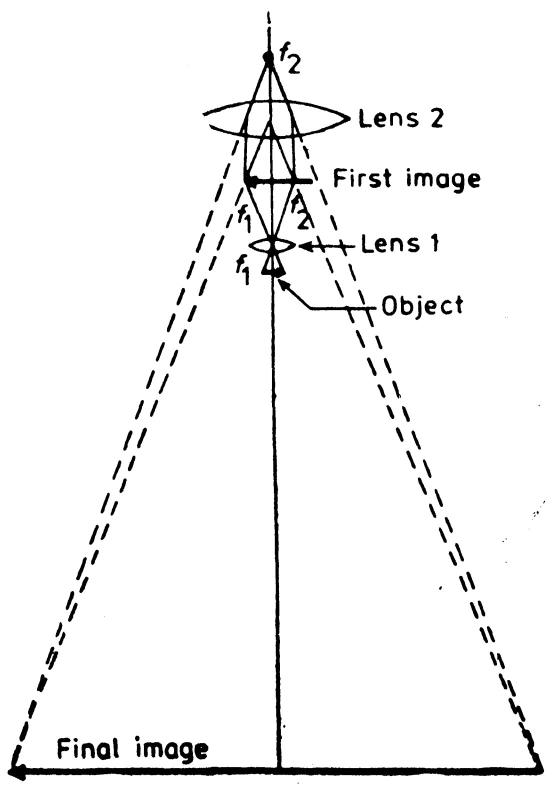
$$m=m_1m_2=(2.3)(10.9)=25.$$

Aliter: The final image distance can be alternatively found from the formula,

$$i_2=\frac{o_1df_2-f_1f_2(d+o)}{(d-f_2)(o-f_1)-of_1}$$

$$=\frac{10\times46\times25-7\times25(46+10)}{(46-25)(10-7)-10\times7}=-243\text{ mm, from lens 2.}$$

The ray diagram is shown in Fig. 42.22.



42.23. (*a*) For L_1

$$o_1 = 20 \text{ cm}$$
$$f_1 = 10 \text{ cm}$$

$$\frac{1}{o_1} + \frac{1}{i_1} = \frac{1}{f_1}$$

$$\frac{1}{20} + \frac{1}{i_1} = \frac{1}{10}$$

whence, $i_1 = 20$ cm.

The image is formed 10 cm from L_2 and is within the focal length f_2. This image acts as an object for L_2.

$$\frac{1}{10} + \frac{1}{i_2} = \frac{1}{12.5}$$

whence, $i_2 = -50$ cm.

The final image is formed on the side of L_2 at distance 50 cm i.e. it coincides with the object.

Overall magnification, $m = m_1 \times m_2 = \frac{i_1}{o_1} \times \frac{i_2}{o_2}$

$$= \left(\frac{20 \text{ cm}}{20 \text{ cm}}\right)\left(\frac{50 \text{ cm}}{10 \text{ cm}}\right) = 5$$

(*b*) The lens system an ay-diagram is show below.

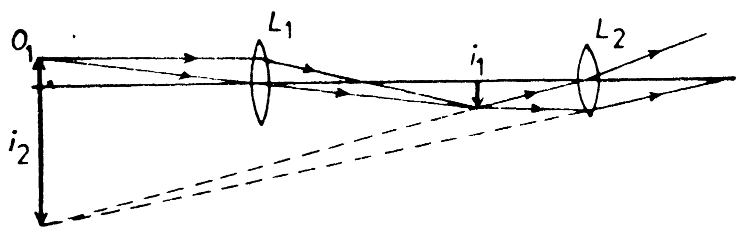


Fig. 42.23

c) The final image is virtual, inverted and magnified.

SUPPLEMENTARY PROBLEMS

S.42.1. Number of images when two plane mirrors are inclined at an angle θ is given by

$$n = \frac{360}{\theta} - 1 = \frac{360}{90} - 1 = 3$$

Due to three combinations, (*i*) wall 1+ceiling (*ii*) wall 2+ceiling (*iii*) wall 1+wall 2, total number of images is $3\times3=9$. Out of these 3 are to be subtracted as they are counted twice, due to three common boundaries. Therefore, total number of images is 6.

S.42.2. (*a*) $\frac{1}{o}+\frac{1}{i}=\frac{1}{f}$ ...(1)

The image will be as far behind the plane mirror as the object is in front.

Set $o=(a+7.5)$ cm

$i=-(a-7.5)$ cm

$f=-30$ cm

$$\frac{1}{a+7.5}-\frac{1}{a-7.5}=\frac{1}{-30}$$

Rearranging and solving for a yields, $a=22.5$ cm.

Ray diagram is shown below in Fig. S.42.2.

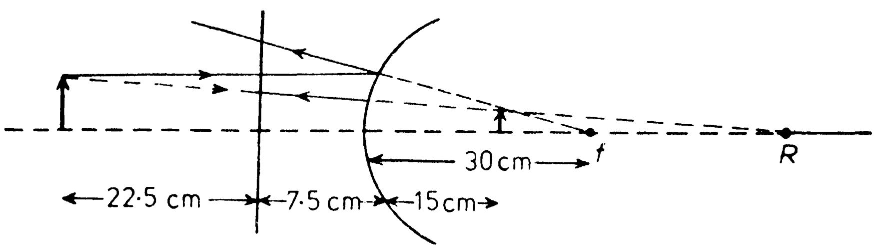


Fig. S.42.2

S.42.3. (*a*) Image $A'B'$ is shown in Fig. S.42.3 at distance $i=D$, the distance of distinct vision.

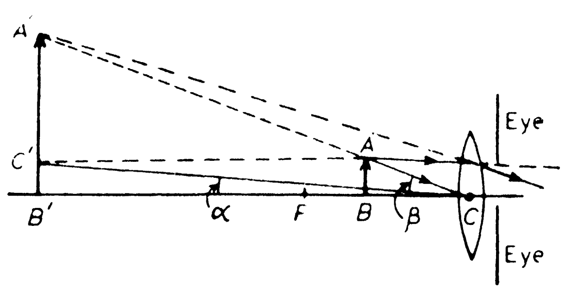


Fig. S.42.3

(*b*) Angular magnification of the simple magnifier lens is defined as

$$M=\frac{\beta}{\alpha}$$ ...(1)

where β is the angle subtended at eye by image at near point and α is the angle that would have been subtended by object placed at near point.

Now, $$\beta=\frac{A'B'}{CB'}=\frac{A'B'}{D} \qquad \ldots(2)$$

and $$\alpha=\frac{C'B'}{CB'}=\frac{AB}{D} \qquad \ldots(3)$$

where $D=25$ cms is the distance of distinct vision.

Now, triangles ABC and $A'B'C$ are similar.

$$\therefore \qquad M=\frac{A'B'}{AB}=\frac{CB'}{CB}=\frac{D}{o} \qquad \ldots(4)$$

where we have used (1), (2) and (3).

As the image is virtual we have

$$\frac{1}{o}-\frac{1}{D}=\frac{1}{f}$$

or $$M=\frac{D}{o}=\frac{D}{f}+1=\frac{25}{f}+1$$

S.42.4. For the combination of two thin lenses of focal length f_1 and f_2 separated by distance d, the image distance i measured from the second lens (f_2) is given by

$$i=\frac{f_2d-f_1f_2\,o/(o-f_1)}{d-f_2-f_1\,o/(o-f_1)}$$

where the object distance is measured from the first lens (f_1). Let the object be placed on the side of the lens with $f_1=12$ cm (converging lens).

$$o=43.5-d/2=(43.5-3.5)\text{ cm}=40\text{ cm}$$

$$f_2=-10\text{ cm and } d=7\text{ cm.}$$

$$i=\frac{(-10)(7)-[(12)(-10)(40)/(40-12)]}{7-(-10)-[(12)(40)/(40-12)]}=-710\text{ cm.}$$

The image is virtual at a distance 710 cm from the second lens ($f_2=-10$ cm) or 713.5 cm from the center of the system on the same side as the converging lens.

Next, let the object be placed on the side of the lens with $f_1=-10$ cm (diverging lens).

Set $f_2=+12$ cm and $o=40$ cm.

$$i=\frac{(12)(7)-[(-10)(12)(40)/(40+10)]}{7-12-[(-10)(40)/(40+10)]}=60\text{ cm}\cdot$$

The image is formed at a distance 60 cm from the converging lens or 63.5 cm from the center of the system on the side of the converging lens.

S.42.5. (*a*) For a combination of two thin lenses,

$$i=\frac{f_2 d-f_1 f_2\, o/(o-f_1)}{d-f_2-f_1 o/(o-f_1)} \qquad \ldots(1)$$

where o and i are the object and image distances as measured in Fig. S.42.5. If we let $o\to\infty$ (incident parallel beam) then the

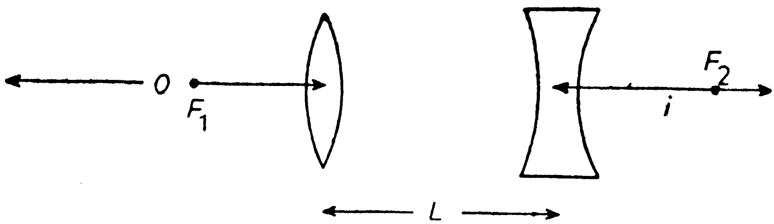


Fig. S.42.5

expression (1) is reduced to

$$i=\frac{f_2(d-f_1)}{d-(f_1+f_2)} \qquad \ldots(2)$$

Setting $f_1=f, f_2=-f$ and $d=L$, formula (2) becomes

$$i=\frac{f(f-L)}{L} \qquad \ldots(3)$$

Formula (3) shows that if i is to be positive L must be less than f. Also d should not be zero. Thus the condition that the parallel beam be brought to a focus beyond the second lens is

$$o<L<f \qquad \ldots(4)$$

(*b*) If the lenses are interchanged then

$f_1=-f\,; f_2=f$ and $d=L$, in which case (2) becomes

$$i=\frac{f(f+L)}{L} \qquad \ldots(5)$$

Here, i is positive irrespective of the distance of separation and the condition (4) need not be satisfied.

(*c*) When $L=0$, in both cases the emerging beam is parallel.

S.42.6. When the two lenses are in contact, the beam of parallel rays remains parallel as it emerges from the other side, (Fig. S.42.6 (*a*). It is as if the beam has fallen on a glass slab. We can look at it in another way. As the beam falls on the concave lens, the rays diverge and appear to come from the focus F. The virtual image at F which is also the focus for the concave lens forms

the object. The rays are therefore rendered parallel due to the convergent lens in contact with the divergent lens. If the lenses are moved apart the virtual focus F of the concave lens will be beyond the focus of the convex lens and the rays upon falling on the convex lens will be brought to focus, the result being independent of the

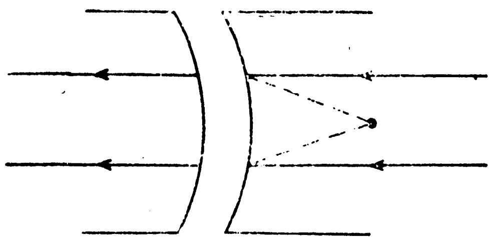


Fig. S.42.6 (a)

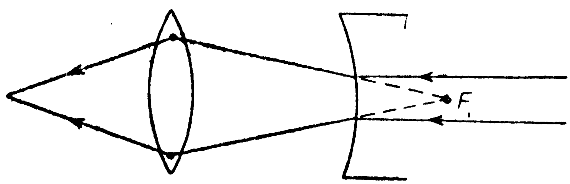


Fig. S.42.6 (b)

distance of separation.

(*b*) When the two lenses are in contact, the rays of a parallel beam upon falling on the convex lens tend to converge at F, the focus of the convex lens. But F is also the virtual focus for the concave lens. Thus, the rays are once again rendered parallel as they emerge from the other side as shown in Fig. S.42.6 (*c*).

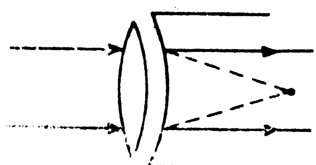


Fig. S.42.6 (c)

If the lenses are moved apart a little then the focus F where the rays ought to have met will be within the virtual focus of the concave lens.

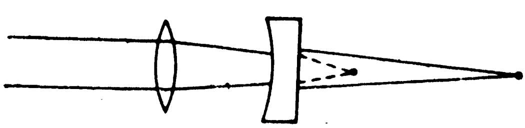


Fig. S.42.6 (d)

The rays will be focussed at a greater distance due to the existence of the diverging lens. This result, however, depends on the

distance of separation of the two lenses. Should this distance be greater than the focal length then the rays fall on the concave lens as divergent rays and owing to the diverging action of the lens will diverge still further.

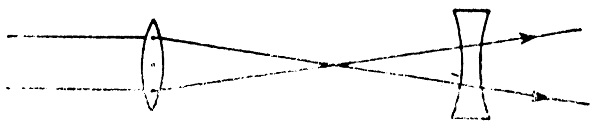


Fig. S.42,6 (e)

S.42.7. (*a*) When the object is placed at distance o_1 in front of convex lens, the image is formed at distance i_1 behind the lens given by

$$\frac{1}{o_1}+\frac{1}{i_1}=\frac{1}{f}$$

$$i_1=\frac{o_1 f}{o_1-f}=\frac{(1\text{ meter})(0.5\text{ meter})}{(1-0.5)\text{ meter}}=1.0\text{ meter}$$

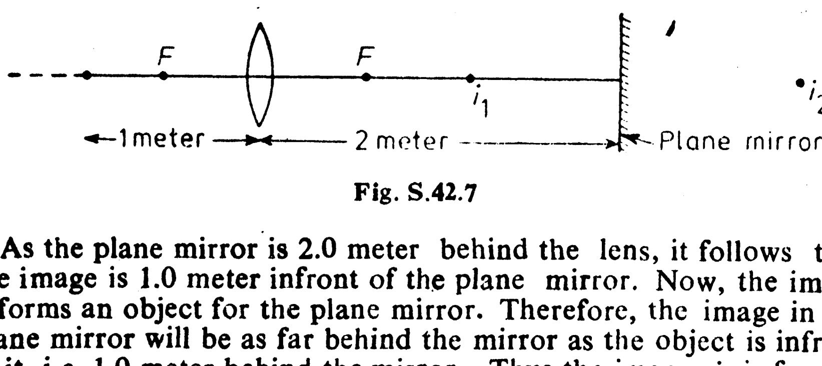


Fig. S.42.7

As the plane mirror is 2.0 meter behind the lens, it follows that the image is 1.0 meter infront of the plane mirror. Now, the image i_1 forms an object for the plane mirror. Therefore, the image in the plane mirror will be as far behind the mirror as the object is infront of it, i.e. 1.0 meter behind the mirror. Thus the image i_2 is formed 3.0 meters behind the convex lens. The image i_2 now acts as an object for the lens.

Set $o_2=3.0$ meter

$$\frac{1}{o_2}+\frac{1}{i_3}=\frac{1}{f}$$

or $$i_3=\frac{o_2 f}{o_2-f}=\frac{(3\text{ meter})(0.5\text{ meter})}{(3.0-0.5)\text{ meter}}=0.6\text{ meter.}$$

Thus the final image is formed 0.6 meter on the side of the lens away from the mirror.

(*b*) The positive sign for i_3 shows that the final image is real.

(*c*) i_1 is inverted and so also i_2. However, i_2 in getting through the lens gets once again inverted. The two inversions cause the final image to be erect.

(*d*) The magnification for the first image i_1 is

$$m_1 = \frac{i_1}{o} = \frac{1.0 \text{ meter}}{1.0 \text{ meter}} = 1$$

In the plane mirror the magnification m_1 is unaltered.

The magnification due to the image i_3 is

$$m_3 = \frac{\text{image distance}}{\text{object distance}} = \frac{0.6 \text{ meter}}{3.0 \text{ meter}} = 0.2 \text{ meter}$$

$\therefore$ Overall magnification $m = m_1 m_2 m_3 = 1 \times 1 \times 0.2 = 0.2$.

43 INTERFERENCE

43.1. The fringe-width for double-slit arrangement is given by

$$\Delta y = \frac{\lambda D}{d}$$

where d is the slit spacing, D is the slit-screen distance and λ is the wavelength of light.

$$\frac{\Delta y}{D} = \theta = \frac{\pi}{180^\circ} = \frac{\lambda}{d}$$

$$\therefore \qquad d = \frac{180\,\lambda}{\pi} = (57.3)(5890 \times 10^{-8}\text{ cm}) = 0.00337 \text{ cm}$$

$$= 0.0337 \text{ mm}.$$

The slits must be 0.0337 mm apart.

43.2. $r_1 - r_2 = 2a$

$$\sqrt{\left(y+\frac{d}{2}\right)^2 + x^2} - \sqrt{\left(y-\frac{d}{2}\right)^2 + x^2} = 2a$$

Transposing one radical, $\sqrt{\left(y+\frac{d}{2}\right)^2 + x^2} = 2a + \sqrt{\left(y-\frac{d}{2}\right)^2 + x^2}$

Squaring and collecting terms,

$$\frac{yd}{2} - a^2 = a\sqrt{\left(y-\frac{d}{2}\right)^2 + x^2}$$

Squaring and simlifying

$$y^2\left(\frac{d^2}{4} - a^2\right) - a^2x^2 = a^2\left(\frac{d^2}{4} - a^2\right)$$

Divide through by $a^2\left(\frac{d^2}{4} - a^2\right)$ to get

$$\frac{y^2}{a^2} - \frac{x^2}{(d^2/4) - a^2} = 1$$

Since $\frac{d}{2} > a$, $\frac{d^2}{4} - a^2$ will be positive.

Writing $(d^2/4) - a^2 = b^2$, we obtain

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

which is an equation of a hyperbola with centre at the origin and the foci on the y-axis.

In three dimensions, the locus of p would be a hyperboloid, the figure of revolution of the hyperbola.

43.3. Angular fringe separation under water.

$$\theta' = \frac{\Delta\theta}{n} = \frac{0.20°}{1.33} = 0.15°.$$

43.4. Angular separation is given by

$$\Delta\theta = \frac{\Delta y}{D} = \frac{\lambda}{d}$$

where D is the slit-screen distance, d is the slit spacing and λ is the wavelength of light.

By Problem, $\Delta\theta = 0.20°$; $\lambda = 5890$ A

New angular separation $\Delta\theta' = 0.22°$, $\lambda' = ?$

$$\lambda' = \lambda \frac{\Delta\theta'}{\Delta\theta} = (5860\ \text{A}°)\left(\frac{0.22°}{0.20°}\right) = 6479\ \text{A}$$

43.5. Shift in fringe system due to insertion of mica flake of thickness t is given by

$$(n-1)\, t = m\lambda$$

or $$t = \frac{m\lambda}{n-1} = \frac{7 \times 5500 \times 10^{-8}\ \text{cm}}{(1.58-1)} = 6.6 \times 10^{-4}\ \text{cm}.$$

43.6. The approximate value for the location of the tenth bright fringe is obtained from

$$y' = \frac{m\lambda D}{d} = \frac{(10)(5890 \times 10^{-8}\ \text{cm})(4\ \text{cm})}{0.2\ \text{cm}} = 0.01178\ \text{cm}.$$

The exact value for y is obtained from the following equations.

$$r_1 = \sqrt{(y'' + d/2)^2 + D^2} \qquad ...(1)$$

$$r_2 = \sqrt{(y'' - d/2)^2 + D^2} \qquad ...(2)$$

$$r_1 - r_2 = m\lambda = \sqrt{(y'' + d/2)^2 + D^2} - \sqrt{(y'' - d/2)^2 + D^2} \qquad ...(3)$$

Transposing one radical

$$\sqrt{\left(y'' + \frac{d}{2}\right)^2 + D^2} = m\lambda + \sqrt{\left(y'' - \frac{d}{2}\right)^2 + D^2}$$

Squaring, collecting terms and solving for y''

$$y'' = m\lambda \sqrt{\left(D^2 + \frac{d^2}{4} - \frac{m^2\lambda^2}{4}\right) / (d^2 - m^2\lambda^2)}$$

$$=\frac{m\lambda D}{d}\sqrt{\left(1+\frac{d^2}{4D^2}-\frac{m^2\lambda^2}{4D^2}\right)\Big/\left(1-\frac{m^2\lambda^2}{d^2}\right)}$$

Since $\frac{m\lambda}{d} \ll 1$,

$$y'' \simeq \frac{m\lambda D}{d}\sqrt{1+\frac{d^2}{4D^2}} = y'\sqrt{1+\frac{(0.2)^2}{(4)(4)^2}} = 1.0003\, y'$$

Fractional error $=(y''-y')/y'=0.0003$.

$\therefore$ Percent error $=0.0003\times 100=0.03$.

43.7. Fringe width, $\triangle y=\lambda D/d$

Set $D=f=1.0$ meter $=100$ cm.

Then, $\triangle y=\dfrac{(5890\times 10^{-8}\text{ cm})(100\text{ cm})}{(0.02\text{ cm})}=0.295\text{ cm}=3.0\text{ mm}$

43.8. (*a*) Condition for maxima at p is

$r_2-r_1=m\lambda$; $m=1,2,3,\ldots$

or $\sqrt{r_1^2+d^2}-r_1=m\lambda$

Set $\lambda=1.0$ meter and $d=4.0$ meter.

Solving, $r_1=\dfrac{16-m^2}{2m}$

(*i*) $m=3$: $r_1=1.16$ meter.

(*ii*) $m=2$; $r_1=3.0$ meter.

(*iii*) $m=1$; $r_1=7.5$ meter.

(*b*) The minimum in intensity along Ox is obtained by setting the path difference

$$\delta=\sqrt{d^2+x^2}-x=(m+\tfrac{1}{2})\lambda \qquad \ldots(1)$$

where $d=4$ meters, $\lambda=1.0$ meter and $m=0,1,2,3\ldots\ldots$ solving for x, we find the minimum value of x for $m=3$, viz., $x=0.53$ meter in which case $\delta=(4.03-0.53)$ meter $=3.5$ meter. Since the path difference is a lot more than the wavelength of the radiation, the conditions approach that of partial coherence rather than complete coherence, and consequently the intensity will not stricity go down to zero i.e. the minimum will not be zero.

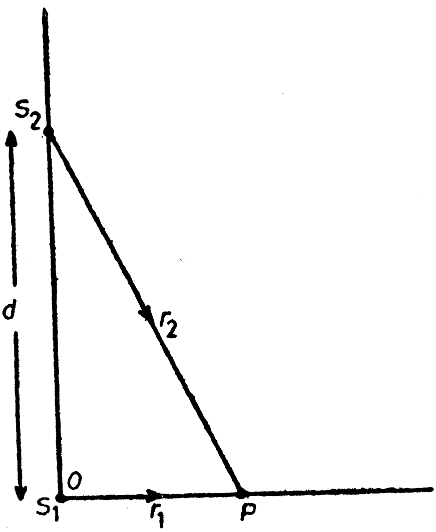


Fig 43.8

43.9. $E_1 = E_0 \sin \omega t$

$$E_2 = 2E_0 \sin(\omega t + \phi)$$

$$E = E_1 + E_2 = E_0 (\sin \omega t + 2 \sin \omega t \cos \phi + 2 \cos \omega t \sin \phi)$$

$$= E_0 [\sin \omega t (1 + 2 \cos \phi) + 2 \cos \omega t \sin \phi]$$

$$= E_0 \sqrt{5 + 4 \cos \phi} \left[\sin \omega t \frac{(1 + 2 \cos \phi)}{\sqrt{5 + 4 \cos \phi}} + \frac{2 \sin \phi}{\sqrt{5 + 4 \cos \phi}} \cos \omega t \right]$$

Set $\cos \alpha = \dfrac{1 + 2 \cos \phi}{\sqrt{5 + 4 \cos \phi}}$

Sin $\alpha = \dfrac{2 \sin \phi}{\sqrt{5 + 4 \cos \phi}}$

$\therefore \quad E = E_0 \sqrt{5 + 4 \cos \phi} (\sin \omega t \cos \alpha + \sin \alpha \cos \omega t)$

$$= E_0 \sqrt{5 + 4 \cos \phi} \sin(\omega t + \alpha)$$

$$= A \sin(\omega t + \alpha)$$

With $A = E_0 \sqrt{5 + 4 \cos \phi}$

$$I_\theta = A^2 = E_0^2 (5 + 4 \cos \phi)$$

$\because \quad I_m = 9E_0^2$

where we have set $\phi = 0$.

$\therefore \quad E_0^2 = \dfrac{I_m}{9}$

$$I_\theta=\frac{I_m}{9}(5+4\cos\phi)=\frac{I_m}{9}[1+4(1+\cos\phi)]$$

$$=\frac{I_m}{9}\left(1+8\cos^2\frac{\phi}{2}\right)$$

But $\dfrac{\phi}{2}=\dfrac{\pi d}{\lambda}\sin\theta$

$\therefore$ $$I_\theta=\frac{I_m}{9}\left[1+8\cos^2\left(\frac{\pi d\sin\theta}{\lambda}\right)\right]$$

43.10. Sin $\theta\simeq\theta=\dfrac{m\lambda}{d}$

$$I_\theta=I_m\cos^2\frac{\pi d\theta}{\lambda}=\frac{I_m}{2}\left(1+\cos\frac{2\pi d\theta}{\lambda}\right)$$

Set $$\frac{I_{\theta'}}{I_m}=\frac{1}{2}$$

where θ' is the angle at which intensity falls to half of maximum value.

Then $\cos\dfrac{2\pi d\theta'}{\lambda}=0$

$\therefore$ $$\frac{2\pi d\theta'}{\lambda}=\frac{\pi}{2}$$

or $$\Delta\theta=2\theta'=\frac{\lambda}{2d}$$

43.11. By constructing the suitable vector diagram, we get
(*a*) $y=y_1+y_2=17$ and $\phi=14°$.

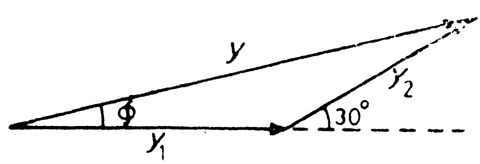


Fig 43.11

(*b*) $$y_1=10\sin\omega t$$
$$y_2=8\sin(\omega t+30°)$$
$$y=y_1+y_2=10\sin\omega t+8\sin(\omega t+30°)$$
$$=10\sin\omega t+8\sin\omega t\cos 30°+8\cos\omega t\sin 30°$$
$$=(10+4\sqrt{3})\sin\omega t+4\cos\omega t$$
$$=A\sin\omega t+B\cos\omega t$$

where $A=10+4\sqrt{3}$ and $B=4$

$$y=\sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\sin\omega t+\frac{B}{\sqrt{A^2+B^2}}\cos\omega t\right)$$

Set $\cos\phi=\frac{A}{\sqrt{A^2+B^2}}$

$\sin\phi=\frac{B}{\sqrt{A^2+B^2}}$

$$y=\sqrt{A^2+B^2}\,(\sin\omega t\cos\phi+\cos\omega t\sin\phi)$$
$$=C\sin(\omega t+\phi)$$

where $C=\sqrt{A^2+B^2}=\sqrt{(10+4\sqrt{3})^2+4^2}=17.39$

$$\phi=\tan^{-1}\frac{B}{A}=\tan^{-1}\left(\frac{4}{10+4\sqrt{3}}\right)=13.3°$$

$\because$ $y=17.39\sin(\omega t+13.3°)$

43.12.

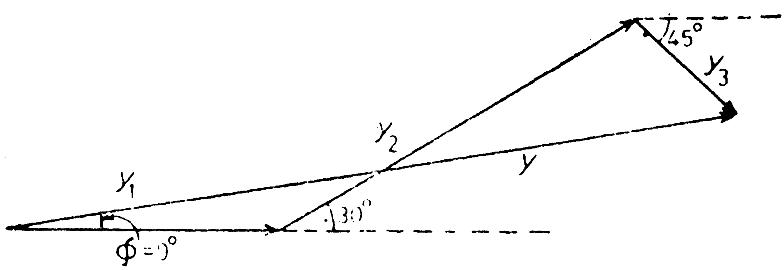


Fig 43.12

$$y=26.6\sin(\omega t+9°)$$

43.13. The visible spectrum extends from 4000° A to 7000° A. Therefore, on the frequency scale, the visible spectrum extends from

$$\frac{3\times10^8\text{ meter/sec}}{4000\times10^{-10}\text{ meter}}\text{ to }\frac{3\times10^8\text{ meter/sec}}{7000\times10^{-10}\text{ meter}}$$

i.e. from 7.5×10^{14} c/s to 4.286×10^{14} c/s.

$\therefore$ Frequency range$=(7.5-4.286)\times10^{14}$, or 3.214×10^{14} c/s

$$\text{Number of channels available}=\frac{\text{frequency range}}{\text{frequency width}}$$

$$=\frac{3.214\times10^{14}\text{ c/s}}{4\times10^6\text{ c/s}}=8\times10^7$$

That is, 80 million

43.14. (*a*) Radius of bright Newton ring,

$$r=\sqrt{(m+\tfrac{1}{2})R\lambda} \quad ...(1)$$

$$m=\frac{r^2}{R\lambda}-\frac{1}{2} \quad ...(2)$$

$$=\frac{(1.0\ \text{cm})^2}{(500\ \text{cm})(5890\times10^{-8}\ \text{cm})}-\frac{1}{2}=33.45$$

or $\quad m=33.$

(*b*) If the arrangement is immersed in water, number of rings that can be seen

$$m'=mn=33.45\times1.33=44.49$$

or $\quad m'=44.$

43.15. $r=\sqrt{(m+\frac{1}{2})\,R\lambda}$

In air, $r_m^2=(m+\frac{1}{2})\,R\lambda$

In liquid $r'^2_m=(m+\frac{1}{2})\,\dfrac{R\lambda}{n}$

$$\therefore \qquad n=\frac{r_m^2}{r'^2_m}=\frac{1.4^2}{1.27^2}=1.21$$

43.16. For bright fringes formed by the air gap of a wedge due to light Incident normally,

$$2\,d=(m+\tfrac{1}{2})\,\lambda$$

where d is the diameter of wire (air gap) at the point the mth fringe is formed

$$m=\frac{2d}{\lambda}-\frac{1}{2}=\frac{(2)(0.0048\ \text{cm})}{(6800\times10^{-8}\ \text{cm})}-\frac{1}{2}=141.$$

43.17. Condition for observing a bright fringe is

$$2nd=(m+\tfrac{1}{2})\,\lambda$$

or $$\lambda=\frac{2nd}{m+\frac{1}{2}}=\frac{2(1.5)(4\times10^{-5}\ \text{cm})}{m+\frac{1}{2}}=\frac{12\times10^{-5}\ \text{cm}}{m+\frac{1}{2}}$$

The integer m which gives the wavelength in the visible region (4000 A to 7000 A) is $m=2$ in which case

$$\lambda=\frac{12\times10^{-5}\ \text{cm}}{2+\frac{1}{2}}=4.8\times10^{-5}\ \text{cm}=4800\ \text{A},$$

which corresponds to blue light.

43.18. For interference maxima,

$$2nd=(m-\tfrac{1}{2})\,\lambda_{max} \qquad \text{...(1)}$$

For intereference minima,

$$2nd=m\,\lambda_{min} \qquad \text{...(2)}$$

Combining (1) and (2),

$$(m-\tfrac{1}{2})(6000\ \text{A})=(m)(4500\ \text{A})$$

whence, $\quad m=2. \qquad$ ...(3)

Use (3) in (2) to get

$$d=\frac{m\lambda_{min}}{2n}=\frac{(2)(4500\ A)}{(2)(1.33)}=3375\ A$$

43.19. Here both the rays suffer a phase change of 180° and the condition for destructive intereference is

$$2nd=(m+\tfrac{1}{2})\ \lambda_1 \qquad ...(1)$$

$$2nd=\left(m+\frac{3}{2}\right)\lambda_2 \qquad ...(2)$$

From (1) and (2)

$$\frac{m+1/2}{m+3/2}=\frac{\lambda_2}{\lambda_1}=\frac{5000\ A}{7000\ A}=\frac{5}{7}$$

whence $\quad m=2.$ $\qquad ...(3)$

Using (3) in (1) to get

$$d=\frac{(m+\frac{1}{2})\ \lambda_1}{2\ n}=\frac{(2.5)(7000\ A)}{(2)(1.3)}=6730\ A.$$

43.20. Phase difference, $\phi=\left(\frac{2\ \pi}{\lambda}\right)(2nd)$

By Problem, for $\lambda=5500$ A, $\phi=\pi$

$\therefore \qquad 2nd=5500$ A

(*i*) $\qquad \lambda_1=4500$ A

$$\phi_1=\left(\frac{2\ \pi}{\lambda_1}\right)(2nd)=\frac{(5500\ A)\ \pi}{(4500\ A)}=\frac{11}{9}\ \pi \text{ radians}=220°$$

$$\frac{I}{I_0}=\cos^2\frac{\phi_1}{2}=\cos^2\left(\frac{220°}{2}\right)=0.117.$$

Reflection intensity is diminished by 0.88 or 88%.

(*ii*) $\qquad \lambda_2=6500$ A

$$\phi_2=\left(\frac{2\ \pi}{\lambda_2}\right)(2nd)=\frac{(5500\ A)\ \pi}{6500\ A}=\frac{11}{13}\pi \text{ radians}=152.3°$$

$$\frac{I}{I_0}=\cos^2\frac{\phi_2}{2}=0.057$$

Reflection intensity is diminished by 0.94 or 94%

43.21. If $n>1.5$, condition for minima is

$$2nd=m\lambda_1 \qquad ...(1)$$

$$2nd=(m+1)\ \lambda_2 \qquad ...(2)$$

where m is an integer.

Combining (1) and (2)

$$(m+1)\ \lambda_2 = m\lambda_1$$

$$(m+1)(5000\ \text{A}) = (m)(7000\ \text{A})$$

whence $m = 2.5$.

This value is unacceptable since m is not an integer.

If $n < 1.5$ condition for minima is

$$2nd = (m+\tfrac{1}{2})\ \lambda_1 \quad \cdot\cdot\ (3)$$

$$2nd = \left(m+\frac{3}{2}\right)\lambda_2 \quad \ldots(4)$$

where m is an integer.

Combining (3) and (4)

$$\left(m+\frac{1}{2}\right)(7000\ \text{A}) = \left(m+\frac{3}{2}\right)(5000\ \text{A})$$

whence, $m = 2$, which is an acceptable value.

Hence, we conclude that the refractive index of oil is less than 1.5.

43.22. Condition for maxima is

$$2nd = (m+\tfrac{1}{2})\ \lambda_{max} \quad \ldots(1)$$

with $\lambda_{max} = 6000$ A and $m = 0, 1, 2, \ldots$

Condition for minima is

$$2nd = (m+1)\ \lambda_{min} \quad \ldots(2)$$

with $\lambda_{min} = 4500$ A corresponding to the violet end of the spectrum and $m = 0, 1, 2, \ldots$

Combining (1) and (2),

$$(m+\tfrac{1}{2})(6000\ \text{A}) = (m+1)(4500\ \text{A})$$

whence, $m = 1$. ...(3)

Use (3) in (1) to find thickness

$$d = \frac{(1.5)(6000\ \text{A})}{(2)(1.33)} = 3383\ \text{A}.$$

43 23. $2d = m\lambda$

$$\therefore \quad \lambda = \frac{2d}{m} = \frac{(2)(0.0233\ \text{cm})}{792} = 5880\times10^{-8}\ \text{cm} = 5880\ \text{A}.$$

43.24. $2(n-1)\ d = m\lambda$

or
$$d = \frac{m\lambda}{2(n-1)} = \frac{7\times5890\ \text{A}}{2(1.4-1)} = 51537\ \text{A}.$$

43.25. (*a*) 6057.8021 A

(*b*) The wavelength of orange-red line of Krypton-86 has been

adopted as a standard of length and the meter is defined as 1,650,763.73 wavelengths in vacuum of this line. Thus, this line is the basic standard of length from which meter is defined and is the reference wavelength in terms of which all other wavelengths of the electromagnetic spectrum are measured.

The question does make since once the meter has been defined as above, it is meaningful to express the wavelengths in terms of meter by way of conversion of units.

43.26. (*a*) Let the wavelengths of sodium light be λ_1 and λ_2. The fringe visibility will be high when the bright bands of λ_1 nearly overlap with those of λ_2, and it will be poor when the bright bands of λ_1 coincide with the dark bands of λ_2. The latter situation arises when the optical path difference is equal to the whole number of wavelengths of λ_1 and an odd number of half-wavelengths of λ_2. Thus, as one of the mirrors is moved, the fringes periodically disappear and then reappear, due to the reason given above and the variation in the visibility is explained.

(*b*) Optical path-difference $=2d=m_1\lambda_1=m_2\lambda_2$, is the condition for a maximum in brightness.

Hence,

$$m_1=\frac{2\,d}{\lambda_1}$$

$$m_2=\frac{2\,d}{\lambda_2}$$

Subtracting, we have

$$m_2-m_1=\frac{2\,d\,\Delta\lambda}{\lambda_1\lambda_2}$$

where $\Delta\lambda=\lambda_1-\lambda_2$

The integer (m_2-m_1) increases by 1 as d changes to $d+\Delta d$, at the next occurrence of maximum brightness. We then have

$$m_2-m_1+1=\frac{2(d+\Delta d)\,\Delta\,\lambda}{\lambda_1\lambda_2}$$

Subtracting the last two equations

$$1=\frac{2\Delta d\;\Delta\lambda}{\lambda_1\;\lambda_2}$$

or $$\Delta\,d=\frac{\lambda_1\lambda_2}{2\,\Delta\lambda}=\frac{(5890\times10^{-8}\text{ cm})(5896\times10^{-8}\text{ cm})}{(2)(5896-5890)\times10^{-8}\text{ cm}}$$

$$=0.029\text{ cm}$$

$$=0.29\text{ mm}.$$

SUPPLEMENTARY PROBLEMS

S.43.1. The position of the fringe of order m on the screen is given by

$$y=m\frac{\lambda D}{d}$$

where d is the separation of the slits and D is the slit-screen distance. The separation of the fringes due to wavelengths λ_1 and λ_2 is

$$\triangle y=\frac{mD}{d}(\lambda_1-\lambda_2)=\frac{(3)(100\text{ cm})(6000-4800)\times 10^{-8}\text{ cm}}{(0.5\text{ cm})}$$

$$=0.0072\text{ cm}=0.072\text{ mm}$$

S.43 2. The fringe width is given by

$$\beta=y_{m+1}-y_m=\frac{\lambda D}{d}$$

$$\therefore \qquad \lambda=\frac{d\beta}{D}=\frac{(12\text{ cm})(18\text{ cm})}{(200\text{ cm})}$$

$$=1.08\text{ cm.}$$

Frequency of vibrations is

$$\nu=\frac{v}{\lambda}=\frac{25\text{ cm/sec}}{1.08\text{ cm}}$$

$$=23.15\text{ cycles/sec.}$$

S.43.3. Shift in the central bright band by n fringes is

$$n=(\mu_2-\mu_1)\ t/\lambda$$

where μ_2 and μ_1 are the refraction indices for the glass plate inserted in the slits and t is the thickness of the plate.

$$t=\frac{n\lambda}{\mu_2-\mu_1}=\frac{(5)(4800\times 10^{-8}\text{ cm})}{(1.7-1.4)}=8\times 10^{-4}\text{ cm}=8\text{ microns.}$$

S.43.4. (*a*) The given arrangement is originally due to Lloyd (1834) and is called Lloyd's mirror. The slit S acts as a source and its reflected rays from the mirror MM' appear to diverge from the virtual image S'. The interference occurs between the rays (SP) directly from S and those ($S'P$) from the virtual image at S'. Owing to the reflection from the mirror, the phase of the reflected ray is increased by π. This leads to the shift in the band system as compared with that in Young's experiment. In particular at the center of the system lies the dark band (minimum) rather than the bright band (maximum).

Drop a perpendicular SQ on $S'P$. Let angle $S'SQ$ be called θ. Then the path difference

$$\delta = S'P - SP = S'P - QP = S'Q = 2\,h \sin\theta$$

Condition for maxima is

$$\delta = 2h \sin\theta = m\lambda + \frac{\lambda}{2} = \left(m + \frac{1}{2}\right)\lambda$$

The additional $\lambda/2$ corresponds to the extra phase difference π which arises due to reflection.

Condition for minima is

$$\delta = 2\,h \sin\theta = m\lambda$$

where m is an integer.

(b) As the reflected rays cannot come below the plane of the surface of mirror, the fringes can appear only in region A.

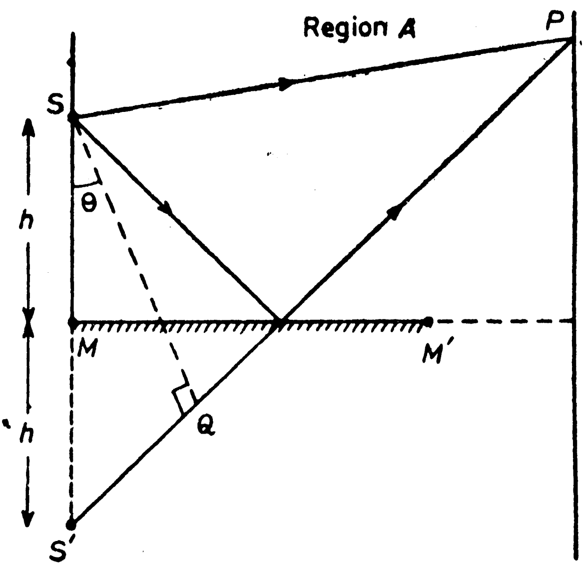


Fig. S.43.4

S.43.5. $[f_1(t)\,f_2(t)]_{av} = \dfrac{1}{T}\displaystyle\int_0^T f_1(t)\,f_2(t)\,dt$

$$= \frac{A_1 A_2}{T} \int_0^T \sin(\omega t + \phi_1)\sin(\omega t + \phi_2)\,dt$$

$$=\frac{1}{2}\frac{A_1A_2}{T}\int_0^T[\cos(\phi_1-\phi_2)-\cos(2\,\omega t+\phi_1+\phi_2)]\,dt$$

where we have used the identity

Sin A sin $B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$

$$\therefore\quad (f_1f_2)_{av}=\frac{1}{2}\frac{A_1A_2}{T}\left[t\cos(\phi_1-\phi_2)-\frac{1}{2\omega}\sin(2\omega t+\phi_1+\phi_2)\right]_0^T$$

The second term in the parenthesis of the right side vanishes when the upper and lower limits are inserted and use is made of $\omega T=2\pi$

$$\therefore\quad (f_1f_2)_{av}=\frac{1}{2}A_1A_2\cos(\phi_1-\phi_2) \qquad \ldots(1)$$

Let the phasors be

$$P_1=A_1\sin(\omega t+\phi_1)$$

and $$P_2=A_2\sin(\omega t+\phi_2)$$

The dot product of the phasors is

$$P_1.P_2=A_1A_2\cos(\phi_1-\phi_2) \qquad \ldots(2)$$

since the angle between the two phasors is $(\phi_1-\phi_2)$. Comparing (1) with (2), we find the desired result

$$(f_1f_2)_{av}=\frac{1}{2}(P_1.P_2).$$

S.43.6. (*a*) Let $S=A_1\sin(\omega\,t+\phi_1)+A_2\sin(\omega\,t+\phi_2)$
$+\ldots+A_n\sin(\omega t+\phi_n)$

$$=A_1(\sin\omega t\cos\phi_1+\cos\omega t\sin\phi_1)$$
$$+A_1(\sin\omega t\cos\phi_2+\cos\omega t\sin\phi_2)$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+A_n(\sin\omega t\cos\phi_n+\cos\omega t\sin\phi_n)$$

$$=\sin\omega t\sum_{i=1}^{n}A_i\cos\phi_i+\cos\omega t\sum_{i=1}^{n}A_i\sin\phi_i$$

$$=B\sin\omega t+C\cos\omega t$$

where $B=\sum_{i=1}^{n}A_i\cos\phi_1$ and $C=\sum_{i=1}^{n}A_i\sin\phi_i$

(*b*) $$B^2+C^2=\sum_{i=1}^{n}A_i^2\cos^2\phi_i+\sum_{i=1}^{n}A_i^2\sin^2\phi$$

$$+2\sum_{i\neq j}\sum_{j}A_iA_j(\cos\phi_i\cos\phi_j+\sin\phi_i\sin\phi_j)$$

$$= \sum_{i=1}^{n} A_i^2 + 2 \sum_{i \neq j} \sum_{j} A_i A_j \cos(\phi_i - \phi_j) \quad \ldots(1)$$

But $$\left(\sum_{i=1}^{n} A_i\right)^2 = \sum_{i=1}^{n} A_i^2 + 2 \sum_{i \neq j} \sum_{j} A_i A_j \quad \ldots(2)$$

Now, the right hand side of (1) is equal or less than the right hand side of (2) because $|\cos(\phi_i - \phi_j)| \leqslant 1$.

$$\therefore \; B^2 + C^2 \leqslant \left(\sum_{i=1}^{n} A_i\right)^2$$

or $$B^2 + C^2 \leqslant (A_1 + A_2 + \ldots A_n)^2 \quad \ldots(3)$$

(*c*) The equality of sign in (3) holds when

$\phi_1 = \phi_2 \ldots\ldots = \phi_n$ that is, all phase angles ϕ_i must be same.

S.43.7. As reflection is from a medium or greater refractive index, we have the conditions,

$$2dn = (m + \tfrac{1}{2})\,\lambda, \; m = 0, 1, 2, \ldots \text{(minimum)} \quad \ldots(1)$$

$$2dn = m\lambda, \; m = 0, 1, 2, \ldots \text{(maximum)} \quad \ldots(2)$$

where d is the thickness of the film and n its refractive index.

$$(2)(1.25)\,d = (6000 \text{ A})(m + \tfrac{1}{2}) \quad \ldots(3)$$

$$(2)(1.25)\,d = (7000 \text{ A})\,m \quad \ldots(4)$$

Combining (3) and (4),

$$(6000 \text{ A})(m + \tfrac{1}{2}) = (7000 \text{ A})\,m$$

or $$m = 3 \quad \ldots(5)$$

Using (5) in (3) or (4), $d = 8400$ A.

S.43.8. Condition for minima is

$$2dn = (m + \tfrac{1}{2})\,\lambda, \; m = 0, 1, 2, \ldots$$

Solving for d,

$$d = \frac{\lambda}{4n}(2m+1) = \frac{(6000 \text{ A})(2m+1)}{(4)(1.25)} = (2m+1)(1200 \text{ A})$$

with $m = 0, 1, 2, \ldots$

S.43.9. Condition for maxima for the film thickness t_1

$$2nt_1 = (m + \tfrac{1}{2})\,\lambda \quad \ldots(1)$$

where m is the order of the fringe.

At a greater thickness t_2 of the film, let the order of the fringe be $m+p$. Then

$$2nt_2 = (m + p + \tfrac{1}{2})\,\lambda \quad \ldots(2)$$

Subtracting (1) from (2),

$$2n(t_2-t_1)=p\lambda$$

or $$t_2-t_1=\frac{p\lambda}{2n}=\frac{(9)(6300\text{ A})}{(2)(1.5)}=18900\text{ A}.$$

As the dark bands are flanked by bright bands, on either side, the information on the number of dark bands is irrelevent to the problem.

S.43.10. (*a*) Here, a phase change of 180° is associated with the rays undergoing reflection both at the upper and lower surface of the oil drop since at both the surfaces the reflection is from a medium of greater refractive index. The thinnest region of the drop must therefore correspond to a bright region. This is in contrast with the dark central spot observed in the system of Newton's rings under reflection where the ray from the bottom of the air film alone under goes a phase change of 180°.

(*b*) Condition for brightness for interference under the given conditions is

$$2nt=m\lambda$$

Setting $\lambda=4750$ A for the blue colour, $n=1.2$ and $m=3$, we get

$$t=\frac{m\lambda}{2n}=\frac{(3)(4750\text{ A})}{(2)(1.2)}=5938\text{ A}.$$

(*c*) Unless the film is reasonably thin i.e. a few wavelengths of light, interference fringes would show up localized on the film. When the oil thickness gets considerably larger, the path difference between the two rays, one getting reflected on the top surface and the other one at the bottom of the film, would amount to several wavelengths, leading to a rapid change in phase difference at a given point as one moves even a small distance away. The interference fringes are blurred out and the pattern disappears.

44 DIFFRACTION

44.1. Condition for getting minima in the diffraction pattern from single slit is

$$a \sin \theta = m\lambda, \; m = 1, 2, 3,$$

where a is the slit-width.

If x is the distance of the minima from the central maxima and D is the slit-screen distance then $\sin \theta \simeq \theta = \dfrac{x}{D}$, provided $x \ll D$.

$$a \simeq \frac{m\lambda}{\theta} = \frac{m\lambda D}{x}$$

But $x = \frac{1}{2}$ (Distance between first minima on either side)

$$= \tfrac{1}{2}\,(0.52 \text{ cm}) = 0.26 \text{ cm}$$

Set $m = 1$ for first minima. Then

$$a = (5460 \times 10^{-8} \text{ cm})(80 \text{ cm})/(0.26 \text{ cm}) = 0.017 \text{ cm}.$$

44.2. $D = f = 70$ cm

(*a*) Linear distance on the screen from the center of the pattern to the first minima is given by setting $m = 1$.

$$x_1 = D \sin \theta \simeq D\,\theta = \frac{D\lambda}{a} = \frac{(70 \text{ cm})(5900 \times 10^{-8} \text{ cm})}{(0.04 \text{ cm})}$$

$$= 0.103 \text{ cm} = 1.03 \text{ mm}$$

(*b*) Linear distance on the screen from the center of the pattern to the second minima is given by setting $m = 2$.

$$x_2 = 2D\theta = \frac{2D\lambda}{a} = (2)(1.03 \text{ mm}) = 2.06 \text{ mm}$$

44.3. (*a*) $\sin \theta = m\lambda$

$a \sin \theta_a = \lambda_a$

$a \sin \theta_b = 2\lambda_b$

(*a*) But $\theta_a = \theta_b$

$\therefore \quad \lambda_a = 2\lambda_b.$

(*b*) $a \sin \theta_a = m_a \lambda_a = 2m_a \lambda_b$

$a \sin \theta_b = m_b \lambda_b$

Set $\theta_a = \theta_b$. Then minima in the two patterns coincide when

$m_b = 2m_a.$

44.4. (*b*) $I_\theta = I_m \left(\dfrac{\sin \alpha}{\alpha}\right)^2$ (Textbook Eq. 44.8b)

Differentiate with respect to α and set

$$\frac{\partial I_\theta}{\partial \alpha} = 0$$

$$\frac{\partial I_\theta}{\partial \alpha} = \frac{I_m\,[2\alpha^2 \sin\alpha \cos\alpha - 2\alpha \sin^2\alpha]}{\alpha^4} = 0$$

or $\quad \alpha \sin\alpha\,(\alpha \cos\alpha - \sin\alpha) = 0$

One solution is $\alpha = 0$. Other solutions are obtained from

$$\alpha \cos\alpha - \sin\alpha = 0$$

or $$\tan\alpha = \alpha$$

(*b*)

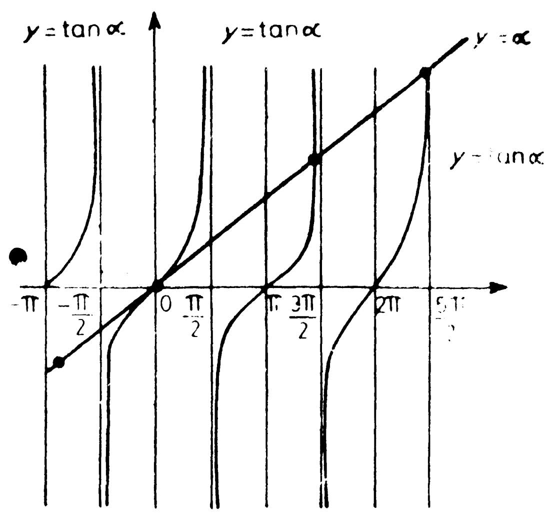


Fig. 44.4

(*c*) We find the first root is zero, then we have a series of roots less than but gradually closer to $(m+\frac{1}{2})\pi = 1, 2, 3\ldots\ldots$ The first three values of m are 0.93, 1.959, 2.971.

44.5. Figure 44.5 shows the graph of $y = \left(\frac{\sin\alpha}{\alpha}\right)^2$. The ordinate y has the value 0.5 for $\alpha = 79°44$.

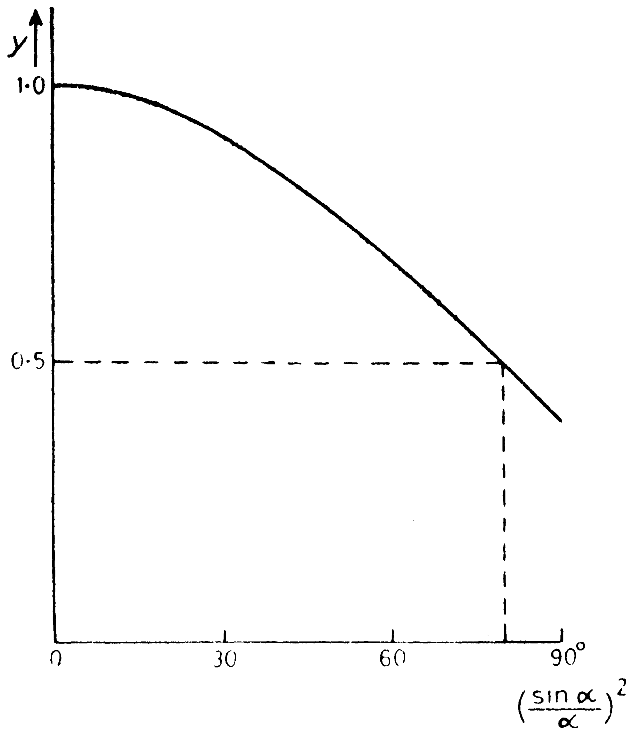


Fig. 44.5

44.6. (*a*) For the first maximum beyond the central maximum since $\phi=2\alpha=2.86\pi$ rather than 3π, the vector $\mathbf{E}_\theta$ is not vertical.

(*b*) The angle which $\mathbf{E}_\theta$ makes with the vertical is given by

$$\frac{(3\pi-2.86\pi)}{3\pi}(540°)=25.2°.$$

44.7. $\sin\theta_x=\dfrac{1.4\lambda}{\pi a}$

(*a*) $a/\lambda=1$

$$\sin\theta_x=\frac{1.4}{\pi}$$

$\therefore$ $\theta_x=26.45°$

Half width, $\Delta\theta=2\theta_x=52.9°$.

(*a*) $a/\lambda=5$

$$\sin\theta_x=\frac{1.4}{5\pi}$$

$$\Delta\theta=(2)(5.1°)=10.2°.$$

(*c*) $a/\lambda = 10$

$$\sin \theta_x = \frac{1.4}{10\pi}$$

$$\Delta\theta = (2)(2.55°) = 5.1°.$$

44.8. (*a*) Condition for obtaining the first minimum from diffraction at a circular aperture is given by

$$\sin \theta = 1.22 \frac{\lambda}{d}$$

where d is the diameter of the aperture, λ the wavelength and θ the angle between the normal to the diaphragm and the direction of the first minimum.

$$\lambda = \frac{v}{\nu} = \frac{1450 \text{ meter/sec}}{25000 \text{ cycle/sec}} = 0.058 \text{ meter}$$

$$\theta = \sin^{-1} \frac{1.22\lambda}{d} = \sin^{-1} \frac{(1.22)(0.058 \text{ meter})}{(0.6 \text{ meter})} = 6.8°$$

(*b*) $$\lambda = \frac{1450 \text{ meter/sec}}{1000 \text{ cycle/sec}} = 1.45 \text{ meter}$$

$$\sin \theta = \frac{(1.22)(1.45 \text{ meter})}{(0.6) \text{ meter})} = 2.948$$

This is impossible as $\sin \theta$ cannot be greater than 1. Hence, the condition for minimum cannot be obtained.

44.9. Rayleigh's, criterion for resolving two objects is

$$\theta_R = \sin^{-1} \frac{1.22\lambda}{d} \simeq \frac{1.22\lambda}{d} = \frac{a}{D}$$

where d is the diameter of the pupil, λ the wavelength, a the distance between the two headlights and D the maximum distance of the automobile at which the eye can resolve the lights.

$$D = \frac{ad}{1.22\lambda} = \frac{(1.22 \text{ meter})(5.0 \times 10^{-3} \text{ meter})}{(1.22)(5500 \times 10^{-10}) \text{ meter})} = 9091 \text{ meters.}$$

44.10. As in Problem 44.9.

$$D = \frac{ad}{1.22\lambda} = \frac{(0.5 \text{ cm})(0.4 \text{ cm})}{(1.22)(5500 \times 10^{-8} \text{ cm})} = 2980 \text{ cm} \simeq 30 \text{ meters.}$$

44.11. (*a*) Angular separation of two stars is given by

$$\theta_R = \frac{1.22\lambda}{d} = \frac{(1.22)(5000 \times 10^{-8} \text{ cm})}{(76.2 \text{ cm})} = 8 \times 10^{-7} \text{ radians}$$

$$= 0.165 \text{ seconds of arc.}$$ 1 sec of arc = ? rad

(*b*) Set $\theta_R = a/D$

where a is the distance between the stars and D is the distance of the stars from the earth.

$D = 10$ light years $= (10)(9.46\times 10^{12}$ km$) = 9.46\times 10^{13}$ km.

$a = D\theta_R = (9.46\times 10^{13}$ km$)(8\times 10^{-7}$ radians$) = 7.57\times 10^{7}$ km.

(*c*) Diameter of the first dark ring is

$$d' = (2)(1.22\, f\lambda)/d = 2\, f\theta_R$$

where f is the focal length of the lens and d' is the diameter of the lens.

$$d' = (2)(1402 \text{ cm})(8\times 10^{-7} \text{ radians}) = 2.24\times 10^{-3} \text{ cm}.$$

44.12. Separation of two points on the moon's surface that can be just resolved is given by

$$a = \frac{1.22\, D\lambda}{d}$$

where D is the distance of moon from the earth, d is the diameter of the lens and λ the wavelength of light that is used.

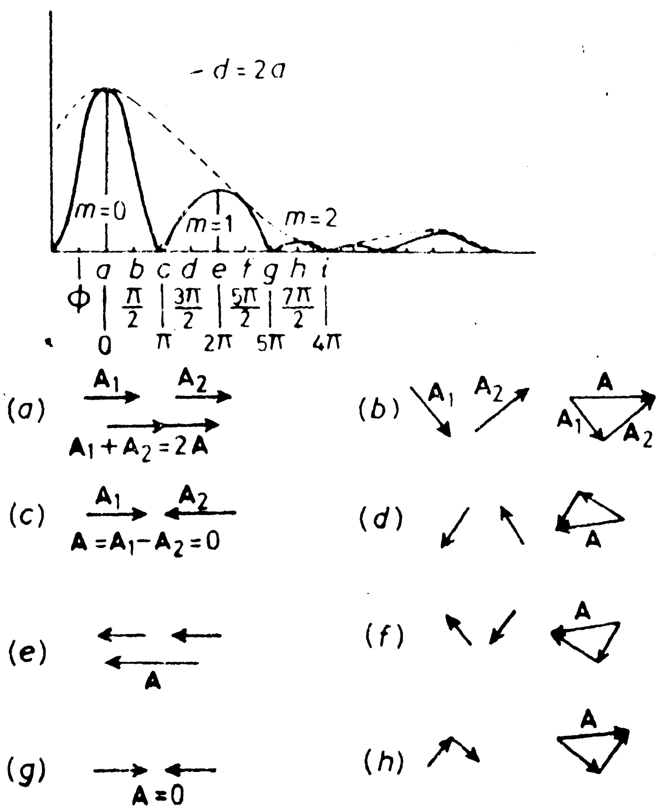
$$D = 240{,}000 \text{ miles} = 3.86\times 10^{8} \text{ meters}$$

$$d = 200 \text{ in} = 5.08 \text{ meters}$$

Let $\quad \lambda = 5500\times 10^{-8} \text{ cm} = 5.5\times 10^{-7}$ meter

Then, $\quad a = \dfrac{(1.22)(3.86\times 10^{8} \text{ meter})(5.5\times 10^{-7} \text{ meter})}{(5.08 \text{ meter})} = 51$ mete

44.13. Diagrams (*a*) to (*h*) in Fig. 44.13 correspond to the points labelled on the intensity curve. The contributions from the two slits



to the amplitudes in magnitude and phase are indicated by A_1 and A_2. The intensity is found as the square of the resultant amplitude A where A is given by the vector addition of $\mathbf{A_1}$ and $\mathbf{A_2}$.

44.14. For the double slit, the combined effect due to interference and diffraction is given by

$$I_\theta = I_m (\cos\beta)^2 \left(\frac{\sin\alpha}{\alpha}\right)^2 \qquad \ldots(1)$$

Condition for minima in the interference pattern is

$$\beta = (m+\tfrac{1}{2})\pi,\ m=0,\ 1,\ 2,\ldots \qquad \ldots(2)$$

where $\beta = \frac{\pi d}{\lambda}\sin\theta$.

Condition for minima in the diffraction is

$$\alpha = \pi,\ 2\pi\ldots \qquad \ldots(3)$$

In Example 8, the 6th minimum of the interference pattern coincides with the first minimum of diffraction pattern.

$$\therefore \qquad \beta = \frac{11\pi}{2} \quad \text{with } m=5$$

$$\alpha = n$$

So, $$\frac{\beta}{\alpha} = \frac{d}{a} = \frac{11}{2}$$

For the second minimum in diffraction pattern, set $\alpha = \pi$.

Then, $\beta = \frac{11}{2}\ \alpha = 11\pi$

From (2), $m=10$ gives $\beta = (10\tfrac{1}{2})\ \pi$ which is the maximum value less than 11π. Thus with $m=0,\ 1,\ 2,\ \ldots\ \ldots 10$, eleven fringes in all are present in the envelope of the central peak. Hence, the number of fringes that lie between the first and the second minima of the envelope is $(11-6)=5$.

44.15. Since $d/a=2$ is an integer, the second order will be missing. There will be one fringe on either side of the central fringe. Hence, in all there will be 3 fringes in the central envelope of the diffraction pattern.

44.16. For the double slit, the combined effect due to interference and diffraction is given by

$$I_\theta = I_m (\cos\beta)^2 \left(\frac{\sin\alpha}{\alpha}\right)^2. \qquad \ldots(44.16)$$

wtth $$\beta = \frac{\pi d}{\lambda}\sin\theta$$

$$\alpha = \frac{\pi a}{\lambda}\sin\theta$$

Set $d=a$. Then $\beta=\alpha$,

$$I_\theta = I_m (\cos\alpha)^2 \left(\frac{\sin\alpha}{\alpha}\right)^2$$

$$= I_m \frac{(2\sin\alpha\cos\alpha)^2}{(2\alpha)^2}$$

$$\therefore \quad I_\theta = I_m \left(\frac{\sin 2\alpha}{2\alpha}\right)^2.$$

The last equation is appropriate for the diffraction pattern of a single slit of width $2a$.

44.17. The missing orders occur where the condition for a maximum of the interference and for a minimum of the diffraction are both fulfilled for the same value of θ. Thus,

$$d \sin\theta = m\lambda, \quad m=0, 1, 2, \ldots$$

$$a \sin\theta = p\lambda, \quad p=1, 2, \ldots$$

so that $$\frac{d}{a}=\frac{m}{p}$$

Since both m and p are integers, d/a must be in the ratio of two integers if the missing orders are going to occur. If order 4 is missing then we must have $d/a=4$.

(*b*) Other fringes which are missing are 8, 12, 16......

SUPPLEMENTARY PROBLEMS

S.44.1. (*a*) The water droplets in the air act as diffracting centres giving rise to the observed rings. The diffraction pattern due to a large number of irregularly arranged circular obstacles is similar to that for a single obstacle.

(*b*) Airy's formula for the intensity distribution for diffraction from a disk of radius a is given by

$$I \propto \left[\frac{J_1(x)}{x}\right]^2 \qquad \ldots(1)$$

where $$x=\frac{2\pi a\alpha}{\lambda} \qquad \ldots(2)$$

and $J_1(x)$ is the Bessel-function of the first order. The position of the first secondary maximum is given by the condition

$$x=5.136 \qquad \ldots(3)$$

Now, the radius of the ring is

$$r=(1.5)(1740)\ \text{km}=2610\ \text{km}.$$

If d is the distance of the moon from earth, the angle θ subtended by the radius of the ring is given by

$$\theta=\frac{r}{d}=\frac{2610 \text{ km}}{38\times 10^4 \text{ km}}=6.9\times 10^{-3} \text{ radians.} \qquad ...(4)$$

Using (3) and (4) in (2) and $\lambda=5.6\times 10^{-5}$ cm for the center of the visible spectrum, the diameter of the water droplets is found from

$$D=2a=\frac{(5.136)(5.6\times 10^{-5} \text{ cm})}{6.9\times 10^{-3}\ \pi}=0.013 \text{ cm}=0.13 \text{ mm.}$$

S.44.2. Two objects like A and B are said to be complementary screens. Let $\mathbf{U}_A$ and $\mathbf{U}_B$ denote respectively the values of the vector amplitude of the light wave when screen A or B alone is placed and let U be the value when no screen is used.

A generalisation known as *Babinet's principle* relates the diffraction patterns produced by two complementary screens. It states that the resultant of the two amplitudes, produced at a point by the two screens, gives the amplitude due to unscreened wave. Thus we may write

$$\mathbf{U}_A+\mathbf{U}_B=\mathbf{U}$$

Setting $\mathbf{U}=0$, we get $\mathbf{U}_A=-\mathbf{U}_B$, showing thereby that at P where $U=0$ the phases of $\mathbf{U}_A$ and $\mathbf{U}_B$ differ by π and the intensities $I_A=|\mathbf{U}_A|^2$ and $I_B=|\mathbf{U}_B|^2$ are equal.

S.44.3. Rayleigh's criterion gives

$$\theta=\frac{x}{d}=1.22\frac{\lambda}{d}$$

where x is the distance between the two point sources, D is the dis tance of the sources from the observer, d is the pupil's diameter.

$$x=1.22\frac{\lambda}{d}D=(1.22)\left(\frac{5500\times 10^{-8} \text{ cm}}{0.5 \text{ cm}}\right)(100\times 5280 \text{ ft})$$
$$=71 \text{ ft.}$$

S.44.4. (*a*) Linear separation of two objects on Mars is given by

$$x=1.22\frac{\lambda}{d}D$$
$$=(1.22)\left(\frac{5500\times 10^{-8} \text{ cm}}{0.5 \text{ cm}}\right)(80.45\times 10^6 \text{ km})$$
$$=1.08\times 10^4 \text{ km.}$$

(*b*) $x=1.22\dfrac{\lambda}{d}D$

$$=(1.22)\left(\frac{5500\times 10^{-8} \text{ cm}}{508 \text{ cm}}\right)(80.45\times 10^6 \text{ km})=10.6 \text{ km.}$$

S.44.5. Assume for simplicity that the sources have equal intensity I_0. Then the intensity pattern is given by,

$$I(\theta)=4I_o\,(\cos\beta)^2\left(\frac{\sin\alpha}{\alpha}\right)^2 \qquad ...(1)$$

with $$\alpha=\frac{\pi a}{\lambda}\sin\theta \qquad ...(2)$$

$$\beta=\frac{\phi}{2}+\frac{\varepsilon_1-\varepsilon_2}{2} \qquad ...(3)$$

and $$\phi=\frac{2\pi d}{\lambda}\sin\theta$$

The term $(\sin\alpha/\alpha)^2$ is due to diffraction and the term $(\cos\beta)^2$ due to interference. The quantity 2β is the phase ϕ due to path difference plus the phase difference of the sources, $(\varepsilon_1-\varepsilon_2)$. If $\varepsilon_1-\varepsilon_2$ is allowed to vary from 0 to 2π, the intensity pattern at a given point goes alternately through maximum and minimum twice.

45 GRATINGS AND SPECTRA

45.1. $d \sin \theta = m\lambda$, where $m=0, 1, 2, \ldots$

Set $\theta = 90°$

$$d = \frac{1}{4000} \text{ cm} = 2.5 \times 10^{-4} \text{ cm}$$

For $\lambda = 4000$ A°

$$m = d \sin \theta / \lambda = \frac{(2.5 \times 10^{-4} \text{ cm})(\sin 90°)}{(4000 \times 10^{-8} \text{ cm})} = 6.25$$

For $\lambda = 7000$ A°,

$$m = \frac{(2.5 \times 10^{-4} \text{ cm}) \times 1}{(7000 \times 10^{-8} \text{ cm})} = 3.57$$

Therefore three complete orders of entire visible spectrum can be produced.

45.2. Let E_0 and ω be the amplitude and the angular frequency of the wave incident on the three-slit grating. Let ϕ be the phase difference between the diffracted waves emerging from S_1 and S_2 and a further ϕ between those from S_2 and S_3. The waves from S_1, S_2 and S_3 are given respectively by

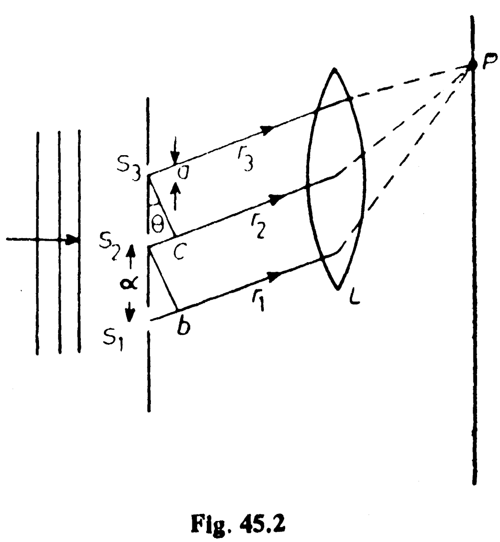


Fig. 45.2

$$E_1 = E_0 \sin \omega t \quad \ldots(1)$$
$$E_2 = E_0 \sin (\omega t + \phi) \quad \ldots(2)$$
$$E_3 = E_0 \sin (\omega t + 2\phi) \quad \ldots(3)$$

with $$\phi = \frac{2\pi d \sin \theta}{\lambda} \quad \ldots(4)$$

Here θ is the angle of diffraction and d is the distance between the centers of two neighbouring slits.

The resultant amplitude at a point P on the screen is given by

$$E = E_1 + E_2 + E_3 = E_0 [\sin \omega t + \sin (\omega t + \phi) + \sin (\omega t + 2\phi)]$$

Expanding the second and third terms in the parenthesis and re-arranging,

$$E = E_0 [\sin \omega t\ (1 + \cos \phi + \cos 2\phi) + \cos \omega t\ (\sin \phi + \sin 2\phi)] \quad \ldots(5)$$

Let the resultant wave at P be given by

$$E = B \sin (\omega t + \gamma) \quad \ldots(6)$$

or $$E = B (\sin \omega t \cos \gamma + \cos \omega t \sin \gamma) \quad \ldots(7)$$

Since (5) and (7) represent the same wave, the coefficients of $\sin \omega t$ and $\cos \omega t$ must be separately equal.

$$B \cos \gamma = E_0 (1 + \cos \phi + \cos 2\phi)$$
$$B \sin \gamma = E_0 (\sin \phi + \sin 2\phi)$$

Squaring both sides, adding and simplifying,

$$B^2 = E_0^2 (1 + 4 \cos \phi + 4 \cos^2 \phi)$$
$$I_m = B^2_{max} = 9E_0^2$$

$$\therefore \quad I_\theta = B^2 = \frac{I_m}{9} (1 + 4 \cos \phi + 4 \cos^2 \phi)$$

45.3. (*a*) Set $$\frac{I_\theta}{I_m} = \frac{1}{2} = \frac{1}{9} (1 + 4 \cos \phi + 4 \cos^2 \phi)$$

or $$\cos^2 \phi + \cos \phi - 7/8 = 0$$

$$\therefore \quad \cos \phi = \frac{3 - \sqrt{2}}{2\sqrt{2}} = 0.56$$

$$\phi = 56^\circ = 0.977 \text{ radians.}$$

Width at half maximum is given by

$$\Delta\theta = 2\theta = \frac{\lambda\phi}{\pi d} = \frac{0.977}{3.14} \frac{\lambda}{d} \simeq \frac{\lambda}{3.2d}$$

(*b*) For two-slit interference frings, $\Delta\theta = \lambda/2d$.

(*c*) Yes.

45.4. $I_\theta \frac{I_m}{9}$ $(1+4\cos\phi+4\cos^2\phi)$ ...(1)

Differentiating with respect to ϕ,

$$\frac{dI_\theta}{d\phi}=-\frac{4}{9}I_m(\sin\phi+2\cos\phi\sin\phi) \quad ...(2)$$

Set $\frac{dI_\theta}{d\phi}=0$

$\sin\phi\,(+2\cos\phi)=0$

The solutions with $0<\phi<2\pi$, are

$$\sin\phi=0 \qquad \text{or } \phi=0,\ \pi,\ 2\pi$$

The correspond to maxima as $\frac{d^2I_0}{d\phi^2}=-\text{ve}$

The other solutions are given by

$$1+2\cos\phi=0$$

$$\therefore \qquad \phi=\frac{2\pi}{3},\ \frac{4\pi}{3}.$$

These correspond to minima as $\frac{d^2I_\theta}{d\phi^2}=+\text{ve}.$

Upon using the values $\phi=0$, π and 2π in (1) we find the intensity as $I_\theta=I_m\ \frac{I_m}{9}$ and I_0 respectively. Of course for the minima $\phi=\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ are expected as $I\theta\ =0$. Thus, the secoedary maximum is located at $\phi=\pi$ and lies half way between the principal maxima, and has intensity $I_m/9$ or $0.11 I_m$.

45.5. As the grating is blazed to concentrate all its intensity in the first order for $\lambda=80{,}000$ A, we have

$$d\sin\theta=m\phi=(1)(80{,}000\times10^{-8}\text{ cm})=8\times10^{-4}\text{ cm}.$$

With the use of visible light (4000 A $<\lambda<$ 7000 A) the order numbers are given by

$$m=\frac{d\sin\theta}{\lambda}=\frac{8\times10^{-4}\text{ cm}}{7\times10^{-5}\text{ cm}}=11 \text{ for red light } (\lambda=7000\text{ A})$$

and $\quad m=\frac{8\times10^{-4}\text{ cm}}{4\times10^{-5}\text{ cm}}=20$, for blue light ($\lambda=4000$ A).

Thus, the intensity is concentrated near the 11th order for red and 20th order for blue light, the orders overlapping to such an extent as to give the appearance of white light for the diffraction beams.

45.6. $\alpha = \dfrac{\phi}{2} = \dfrac{\pi a}{\lambda} \sin\theta$

where ϕ is the phase difference between rays from the top and bottom of the slit and the path difference for these rays is $a \sin\theta$.

For first minimum, $\alpha = \pi$. Also, set $a = Nd$,

where N is the number of slits and d is the grating element. Replace θ by $\Delta\theta_0$, then,

$$\alpha = \pi = \frac{\pi N d \sin \Delta\theta_0}{\lambda}$$

$$\sin \Delta\theta_0 = \frac{\lambda}{Nd}$$

where the angle $\Delta\theta_0$ corresponds to zero intensity that lies on either side of the central principal maximum. For actual grantings, $\sin\Delta\theta_0$ will be quite small so that we can replace $\sin\Delta\theta_0$ by $\Delta\theta_0$ to a good approximation and obtain the desired result

$$\Delta\theta_0 = \frac{\lambda}{Nd}$$

45.7. The grating element, $d = \dfrac{2.54 \text{ cm}}{8000} = 3.175 \times 10^{-4}$ cm

Condition for maxima is

$$d \sin\theta = m\lambda, \quad m = 0, 1, 2 \ldots$$

For the maximum wavelength that can be observed in the fifth order set $\theta = 90$ and $m = 5$. Then,

$$\lambda = \frac{3.175 \times 10^{-4} \text{ cm}}{5} = 6.350 \times 10^{-5} \text{ cm or } 6350 \text{ A}$$

Thus, for all wavelengths shorter than 6350 A the diffraction in the fifth order can be observed.

45.8. (*a*) The grating element $d = \dfrac{2.54}{5000}$ cm $= 5.08 \times 10^{-4}$ cm

Diffracted light at angle $\theta = 30°$ is observed such that

$$d \sin\theta = m\lambda$$

or $\quad m\lambda = (0.5)(5.08 \times 10^{-4} \text{ cm}) = 2.54 \times 10^{-4}$ cm.

In order that λ may be in the visible region (4000 A $< \lambda <$ 7000 A), the corresponding diffraction orders and the associated wavelengths are given as follows:

$m=4$; $\lambda=6350$ A (red)
$m=5$; $\lambda=5080$ A (green)
$m=6$; $\lambda=4233$ A (blue)

(*b*) The wavelengths can be identified by observing their colours

45.9. The grating element, $d=\dfrac{2.0}{6000}$ cm $=3.33\times10^{-4}$ cm

$$d \sin\theta = m\lambda$$

$$\therefore \qquad \sin\theta = m\frac{\lambda}{d} = m\left(\frac{5890\times10^{-8}\text{ cm}}{3.33\times10^{-4}\text{ cm}}\right)=0.1767m$$

As $\sin\theta$ cannot exceed unity, the order of diffraction m is restricted to 0, 1, 2, 3, 4 and 5.

The corresponding angles of diffraction are then,

$$\theta_0=\sin^{-1} 0$$
$$\theta_1=\sin^{-1}(0.1767\times1)=10^\circ$$
$$\theta_2=\sin^{-1}(0.1767\times2)=21^\circ$$
$$\theta_3=\sin^{-1}(0.1767\times3)=32^\circ$$
$$\theta_4=\sin^{-1}(0.1767\times4)=45^\circ$$
$$\theta_5=\sin^{-1}(0.1767\times5)=62^\circ$$

45.10. $d\sin\theta=m\lambda$

Set $\quad m=1$

For $\quad \phi=4300$ A

$$d\sin\theta_1=(1)(4300\times10^{+8}\text{ cm})=4.3\times10^{-5}\text{ cm} \qquad \ldots(1)$$

For $\quad \lambda=6800$ A

$$d\sin\theta_2=(1)(6800\times10^{-8}\text{ cm})=6.8\times10^{-5}\text{ cm} \qquad \ldots(2)$$

By Problem, $\theta_2-\theta_1=20^\circ \qquad \ldots(3)$

Adding (1) and (2),

$$2d\sin\frac{\theta_2+\theta_1}{2}\cos\frac{\theta_2-\theta_1}{2}=11.1\times10^{-5}\text{ cm} \qquad \ldots(4)$$

Subtracting (2) from (1),

$$2d\cos\frac{\theta_2+\theta_1}{2}\sin\frac{\theta_2-\theta_1}{2}=2.5\times10^{-5}\text{ cm} \qquad \ldots(5)$$

Dividing (4) by (5)

$$\tan\frac{\theta_2+\theta_1}{2}\cot\frac{\theta_2-\theta_1}{2}=4.44 \qquad \ldots(6)$$

Use (3) in (6) to find

$$\tan\frac{\theta_2+\theta_1}{2}=4.44, \text{ and } \tan\frac{\theta_2-\theta_1}{2}=4.44\tan 10^\circ=0.7828.$$

where use has been made of (3).

Hence, $\theta_2+\theta_1=76^\circ$ ...(7)

Solve (3) and (7) to find

$$\theta_1=28^\circ\ ;\ \theta_2=48^\circ. \qquad ...(8)$$

Use (8) in (1) or (2) to get

$$d=\frac{4.3\times 10^{-5}\text{ cm}}{\sin 28^\circ}=9.168\times 10^{-5}\text{ cm}.$$

$\therefore$ Number of lines per cm, $N=\dfrac{1}{d}=\dfrac{1}{9.168\times 10^{-5}\text{ cm}}=10900$.

Number of lines per inch $=10900\times 2.54=27700$.

45.11. The grating element, $d=\dfrac{2.54\text{ cm}}{8000}=3.17\times 10^{-4}$ cm.

The angular range through which the spectrum can pass is defined by θ_1 and θ_2 (Fig. 45.11).

$$\tan\theta_1=\frac{5\text{ cm}}{30\text{ cm}}=0.1667$$

$$\therefore \quad \theta_1=9^\circ\ 28'$$

$$\sin\theta_1=0.1648$$

$$d\sin\theta_1=m\lambda_1$$

$$\lambda_1=(3.17\times 10^{-4}\text{ cm})(0.1648)$$

$$=5220\times 10^{-8}\text{ cm}=5220\text{ A}$$

$$\tan\theta_2=\frac{6\text{ cm}}{30\text{ cm}}=0.2$$

$$\theta_2=11^\circ\ 18'$$

$$\therefore \sin\theta_2=0.196$$

$$d\sin\theta_2=m\lambda_2$$

$$\lambda_2=(3.17\times 10^{-4}\text{ cm})(0.196)$$

$$=6210\times 10^{-8}\text{ cm}=6210\text{ A}$$

where $m=1$.

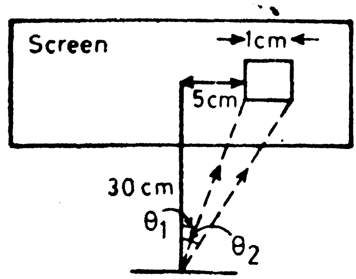


Fig. 45.11

Thus, the range of wavelengths that pass through the hole is 5220A to 6210 A.

45.12. The path difference between adjacent rays like 1 and 2 is

$$\delta = BC + CD = d \sin \psi + d \sin \theta$$
$$= d (\sin \psi + \sin \theta)$$

Condition for obtaining diffraction maxima is

$$\delta = m\lambda, \ m = 0, 1, 2, \ldots$$

i.e. $d(\sin \psi + \sin \theta) = m\lambda, \ m = 0, 1, 2, \ldots$

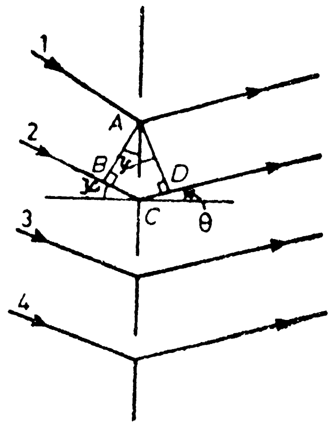


Fig. 45.12

45.13. Maxima in the interference pattern occur when

$$d \sin \theta = m\lambda$$

where $\quad m = 0, 1, 2, \ldots$

Minima in the diffraction pattern result if

$$a \sin \theta = m'\lambda$$

where now $m' = 1, 2, \ldots$

When both the conditions are satisfied for the same angle θ,

$$\frac{d}{a} = \frac{m}{m'} = M$$

If alternately transparent and opaque strips of equal width are used in the grating then $d = 2a$ and consequently $M = 2$. Therefore, all the even orders $m = 2, 4, 6, \ldots$(except $m = 0$) will be missing.

45.14. For various values of angle of incidence ψ ranging from 0° to 90°, we calculate the angle of diffraction θ using the grating equation,

$$d(\sin \psi + \sin \theta) = m\lambda$$

with $m = 1$ for the first order, $\lambda = 6 \times 10^{-5}$ cm and $d = 1.5 \times 10^{-4}$ cm. The angle of deviation $\delta = \psi \pm \theta$, the +ve sign is taken when the incident and diffracted light is on the same side of the normal and −ve when on opposite side.

Fig. 45.14 shows the plot of δ versus ψ.

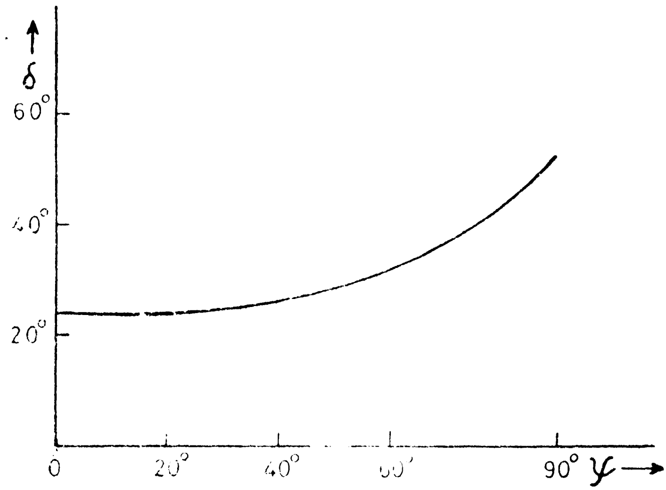


Fig. 45.14

45.15. The principal maxima occur when

$$d \sin \theta = m\lambda.$$

The path difference between the light which gives rise to principal maximum and the immediate minimum is λ/N, where N is the number of slits. Let a change of angle $\Delta\theta$ produce this path difference. Then

$$d \sin(\theta + \Delta\theta) - d \sin \theta = \lambda/N$$

$$d(\sin \theta \cos \Delta\theta + \cos \theta \sin \Delta\theta - \sin \theta) = \lambda/N$$

For small angles, $\sin \Delta\theta \to \Delta\theta$ and $\cos \Delta\theta \to 1$.

Hence,

$$d \cos \theta \Delta\theta = \lambda/N$$

or $$\Delta\theta = \lambda/Nd \cos \theta$$

Thus for order m,

$$\Delta\theta_m = \lambda/Nd \cos \theta.$$

45.16. The half-width of the fringes for a three-slit diffraction pattern is given by

$$\Delta \theta \simeq \lambda/3.2\ d. \quad (1)$$

The half-width of the double-slit interference fringes is given by

$$\Delta\theta = \lambda/2d \quad ...(2)$$

If the middle slit in the three-slit grating is closed then it reduces to a double-slit and the grating element beomes $2d$. Replacing d by $2d$ in (2),

$$\Delta\theta = \lambda/4d \quad ...(3)$$

Comparing (3) with (1), we find that the half width of the intensity maxima has become narrower by 20%.

45.17. (*a*) The grating element, $d=\dfrac{(3)(2.54 \text{ cm})}{40{,}000}=1.9\times10^{-4}$ cm

$$=1.9\times10^{4} \text{ A}$$

$$d \sin\theta=m\lambda$$

For 1st order $\theta_1=\sin^{-1}\dfrac{\lambda}{d}=\sin^{-1}\left(\dfrac{5890 \text{ A}}{1.9\times10^{4} \text{ A}}\right)=\sin^{-1} 0.31=18°.$

Dispersion $D=\dfrac{km}{d\cos\theta}$

$$=\frac{1}{(1.9\times10^{4} \text{ A})\cos 18°}=0.553\times10^{-4} \text{ radians/A}$$

$$=3.17\times10^{-3} \text{ deg/A.}$$

For 2nd order, $\theta_2=\sin^{-1}\dfrac{2\lambda}{d}=\sin^{-1}\dfrac{2\times5890 \text{ A}}{1.9\times10^{4} \text{ A}}=38.2°.$

$$D=\frac{2}{(1.9\times10^{4} \text{ A})\cos 38.2°}=1.34\times10^{-4} \text{ radians/A}$$

$$=7.67\times10^{-3} \text{ deg/A.}$$

For the 3rd order, $\theta_3=\sin^{-1}\dfrac{3\times5890 \text{ A}}{1.9\times10^{4} \text{ A}}=\sin^{-1} 0.93=68.5°$

$$D=\frac{3}{(1.9\times10^{4} \text{ A})\cos 68.5°}$$

$$=4.31\times10^{-4} \text{ radians/A}=24.7\times10^{-3} \text{ deg/A.}$$

(*b*) The resolving power is given by $R=Nm$, where N is the number of rulings and m is the order.

For the 1st order, $R=40{,}000\times1=40{,}000$

For the 2nd order, $R=40{,}000\times2=80{,}000$

For the 3rd order, $R=40{,}000\times3=120{,}000$

45.18. $\Delta\lambda=5895.9 \text{ A}-5890 \text{ A}=5.9 \text{ A}$

$m=3$

$\theta=80°$

Mean wavelength $\bar{\lambda}=\frac{1}{2}(5890+5895.9) \text{ A}=5893 \text{ A}$

$$d\sin\theta=m\lambda$$

(*a*) Grating spacing, $d=\dfrac{m\lambda}{\sin\theta}=\dfrac{(3)(5893\times10^{-8} \text{ cm})}{\sin 80°}$

$$=17952\times10^{-8} \text{ cm}=18000 \text{ A.}$$

(*b*) Resolving power, $R=\frac{\lambda}{\Delta\lambda}=Nm$

$\therefore$ Number of rulings, $N=\frac{\lambda}{m\ \Delta\lambda}=\frac{5893\ \text{A}}{(3)(5.9\ \text{A})}=333$

$\therefore$ Total width of the rulings, $Nd=(333)(18000\ \text{A})=6\times10^6\ \text{A}$
$=0.6$ mm

45.19. $N=\frac{\lambda}{m\ \Delta\lambda}=\frac{6563\ \text{A}}{(1)(1.8\ \text{A})}=3646$ lines.

45.20. $d\sin\theta=m\lambda$

$$\frac{d\theta}{d\lambda}=\frac{m}{d\cos\theta}$$

Equation of grating is

$d\sin\theta=m\lambda$.

$$\therefore \quad \frac{m}{d}=\frac{\sin\theta}{\lambda}$$

$$\therefore \quad D=\frac{d\theta}{d\lambda}=\frac{\sin\theta}{\lambda\cos\theta}=\frac{\tan\theta}{\lambda}$$

45.21. Number of rulings, $N=\left(\frac{6000}{\text{cm}}\right)(6\ \text{cm})=36000$

(*a*) $\qquad R=\frac{\lambda}{\Delta\lambda}=Nm$

$$\Delta\lambda=\frac{\lambda}{Nm}=\frac{5000\text{A}}{(36000)(3)}=0.046\text{A}.$$

(*b*) Resolution is generally improved by going over to higher order m for a given grating (N fixed) and given λ. However, in the present case $m=3$ is the highest order that can be employed for normal incidence.

45.22. In order that dispersion be as high as possible (condition 2) maximum value for the diffraction angle must be chosen. Since the first and second maxima are restricted to angles upto 30°, (Condition 1), set

$m=2$ and $\theta=30°$ in the grating equation,

$d\sin\theta=m\lambda$

(*a*) $\therefore \quad d=\frac{m\lambda}{\sin\theta}=\frac{(2)(6000\text{A})}{0.5}$ 24,000A

where we have used the value of the higher wavelength.

(*b*) Since the third order is missing we have

$$\frac{d}{a}=3$$

where a is the slit width.

$$\therefore \qquad a=\frac{d}{3}=\frac{1}{3}(24000 \text{ A})=8000 \text{ A}$$

(*c*) The maximum order that can actually be seen is given by setting $\theta=90°$ and with the choise of $\lambda=6000$ A.

$$d \sin \theta=m\lambda$$

$$m=\frac{d \sin \theta}{\lambda}=\frac{(24000 \text{ A})(\sin 90°)}{6000 \text{ A}}=4.$$

The orders that actually appear on the screen are $m=1$ and 2. since $m=3$ is missing and $m=4$ occurs at $\theta=90°$.

45.23. (*a*) Grating equation is, $d \sin \theta=m\lambda$.

For order m and angle θ_1 we have

$$d \sin \theta_1=m\lambda \qquad \ldots(1)$$

whilst for order $(m+1)$ and angle θ_2,

$$d \sin \theta_2=(m+1)\lambda \qquad \ldots(2)$$

Dividing (2) by (1)

$$\frac{m+1}{m}=\frac{\sin \theta_2}{\sin \theta_1}=\frac{0.3}{0.2}$$

whence $m=2$.

Using (3) in (1), we find the separation between adjacent slits,

$$d=\frac{m\lambda}{\sin \theta_1}=\frac{(2)(6000 \text{ A})}{0.2}=6\times 10^4 \text{ A}.$$

(*b*) Since the fourth order is missing, we have

$$\frac{d}{a}=4$$

where a is the slit-width.

$\therefore$ Smallest possible slit-width, $a=\dfrac{d}{4}=\dfrac{6\times 10^4 \text{ A}}{4}=1.5\times 10^4$ A.

(*c*) The maximum possible order is obtained by setting $\theta=90°$ in the grating equation

$$d \sin \theta=m\lambda$$

$$\therefore \qquad m=\frac{d \sin \theta}{\lambda}=\frac{(6\times 10^4 \text{ A})(\sin 90°)}{(6000 \text{ A})}=10.$$

Also, since $m=4$ is missing, it follows that $m=8$ will also be missing.

The orders actually appearing on the screen are $m=0, 1, 3, 5, 6, 7, 9$. The tenth order occurs at $\theta=90°$.

45.24. For oblique incidence the grating equation is

$$d(\sin\psi-\sin\theta)=m\lambda \qquad \text{..(1)}$$

where ψ is the angle of incidence, θ the angle of diffraction and m the order of diffraction. We have assumed that the diffracted beam and the incident beams are on the opposite sides of the normal.

$$\psi=90-\gamma \qquad \text{...(2)}$$

$$\theta=90-2\beta \qquad \text{...(3)}$$

$$m=1 \qquad \text{...(4)}$$

Use (2), (3) and (4) in (1) to find

$$d\,[\sin 90-\gamma)-\sin(90-2\beta)]=\lambda$$

or $$\cos\gamma-\cos 2\beta=\lambda/d$$

$\therefore$ $$2\sin\tfrac{1}{2}(\gamma+2\beta)\sin\tfrac{1}{2}(2\beta-\gamma)=\lambda/d$$

or $$\sin\left(\beta+\frac{\gamma}{2}\right)\sin\left(\beta-\frac{\gamma}{2}\right)=\frac{\lambda}{2d}$$

As γ is small one expects β also to be small.

Replacing since function by the argument

$$\left(\beta+\frac{\gamma}{2}\right)\left(\beta-\frac{\gamma}{2}\right)=\frac{\lambda}{2d}$$

which upon simplification and re-arrangement yields the result

$$\beta=\sqrt{\frac{\lambda}{2d}+\frac{\gamma^2}{4}}$$

By Problem, $$\frac{\lambda}{d}=\frac{5A}{15000\text{ A}}=\frac{1}{3000}.$$

Therefore $$\beta=\sqrt{\frac{1}{6000}+\frac{\gamma^2}{4}}$$

45.25. Bragg's law is

$$2d\sin\theta=m\lambda,\ m=1, 2, 3, \ldots$$

$$d=2.81\text{ A}$$

For $$m=1,\ \sin\theta=\frac{\lambda}{2d}=\frac{1.2\text{A}}{(2)(2.81\text{A})}=0.213$$

$\therefore$ $$\theta=12.3°$$

Hence, the angle through which the crystal must be rotated is (45°−12.3°) or 32.7° in the clockwise direction.

For $\quad m=2,\ \sin\theta=\dfrac{(2)(1.2\ \text{A})}{(12)(2.81\ \text{A})}=0.427$

$\therefore \quad \theta=25.3°$

The crystal must be rotated through an angle (45°−25.3°) or 19.7° in the clockwise direction.

For $\quad m=3,\ \sin\theta=\dfrac{(3)(1.2\ \text{A})}{(2)(2.81\ \text{A})}=0.64$

$\therefore \quad \theta=39.8°$

The crystal must be rotated through an angle (45°−39.8°) or 5.2° in the clockwise direction.

For $\quad m=4,\ \sin\theta=\dfrac{(4)(1.2\ \text{A})}{(2)(2.81\ \text{A})}=0.854$

$\therefore \quad \theta=58.6°$

∴ The crystal must be rotated through an angle (58.6°−45°) or 13.6° in the counter clockwise direction. Higher order reflection is not possible as sin θ would exceed 1.

45.26. The sodium chloride crystal has face-centered cubic lattice. The basis consists of one Na-atom and one Cl-atom. There are four units of NaCl in each unit cube, the positions of the atoms being:

Na $\quad 000\ ;\ \tfrac{1}{2}\tfrac{1}{2}0\ ;\ \tfrac{1}{2}0\tfrac{1}{2}\ ;\ 0\tfrac{1}{2}\tfrac{1}{2}$

Cl $\quad \tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\ ;\ 00\tfrac{1}{2}\ ;\ 0\tfrac{1}{2}0\ ;\ \tfrac{1}{2}00$

Each atom has in its neighbourhood six atoms of the other kind. The reflection from an atomic plane through the top of a layer of unit cells is canceled by a reflection from a plane through middle of this layer of cells because a phase difference of π rather than 2π is introduced.

45.27. Bragg's law is

$$2d\sin\theta=m\lambda$$

$$\lambda=\frac{(2)(2.75\ \text{A})(\sin 45°)}{m}=\frac{3.889\ \text{A}}{m}$$

For $\quad m=3,\ \lambda=1.296$ A.

For $\quad m=4,\ \lambda=0.972$ A.

Thus, diffracted beams of wavelength 1.29 A and 0.97 A occur.

45.28. Bragg's law is

$$2d \sin \theta = m\lambda$$

For line B, $d = \frac{m\lambda}{2 \sin \theta} = \frac{(3)(0.97\text{A})}{(2)(\sin 60°)} = 1.68$ A.

For line A, $\lambda = \frac{2d \sin \theta}{m} = \frac{(2)(1.68\text{A}) \sin 30°}{1} = 1.68$ A.

SUPPLEMENTARY PROBLEMS

S.45.1.

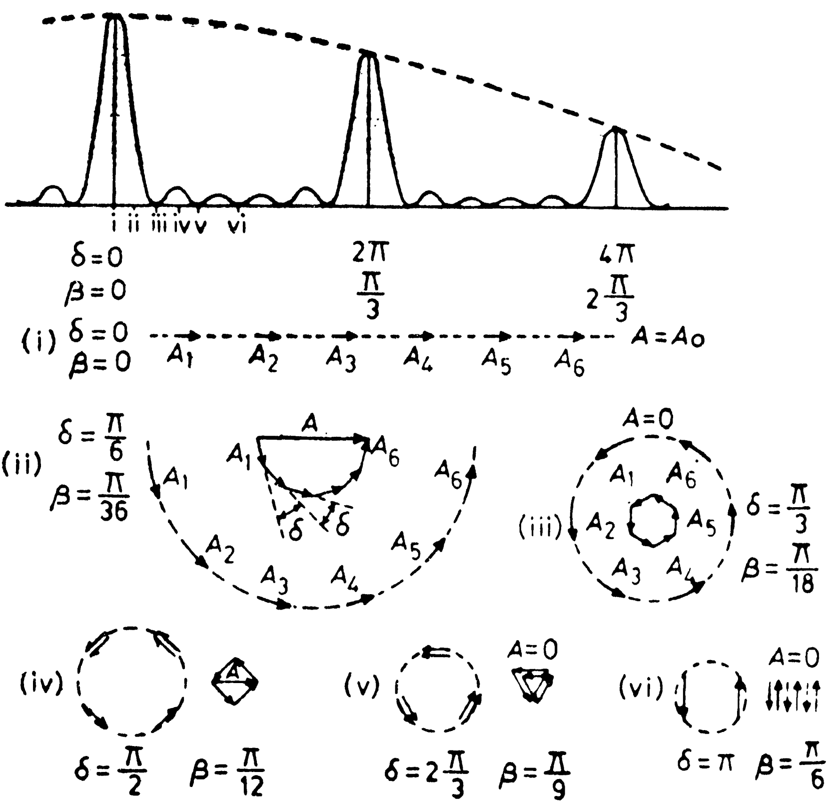


Fig. 44.13

S.45.1. In Fig S.45.1 (*a*) the diagrams corresponding to the points (*a*) to (*f*) of the intensity plot for six slits are shown. For the central maximum light is in phase from all slits as well as that from

a single slit as in (*i*) of the figure and gives resultant amplitude A which is N times as large as from a single slit. In (*ii*) is shown the condition half way to the first minimum. This point corresponds to $\alpha=\pi/12$ so that the phase difference from corresponding points in adjacent slits is $\delta=\pi/6$, this being also the angle between successive vectors $\mathbf{A}_1$ to $\mathbf{A}_6$. The resultant amplitude $\mathbf{A}$ is given by compounding these vectorially and the intensity is given by A^2.

For the derivation of the general intensity formula, consider Fig. S.45.1 (*b*) wherein are shown the six amplitude vectors with phase difference slightly less than in (*ii*) of Fig. S.45.1 (*a*) The magnitude of each of these is identical and is given by

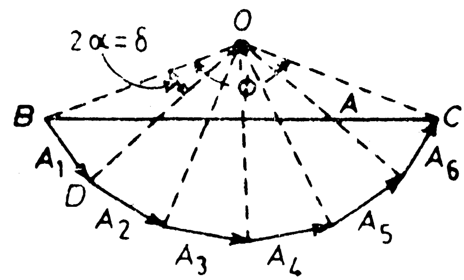


Fig. S.45.1 (b)

$$A_n = \frac{(\sin \beta)}{\beta} A_o \qquad \ldots(1)$$

The amplitude A_n is represented by the chord of an arc of length A subtending an angle 2β at the centre (single slit diffraction pattern). Successive vectors are inclined to each other by an angle $\delta=2\alpha$. Also each of the vectors subtends a constant angle 2α at the center, indicated by the broken lines in the figure. The six vectors form a part of a polygon with center at O, the total angle subtended at O being

$$\phi=N\delta=N(2\alpha) \qquad \ldots(2)$$

From the triangle OBC, we find that the resultant amplitude is

$$A=2r \sin \phi/2 \qquad \ldots(3)$$

where $r=OB$ is the radius of the inscribed circle. Similarly, from the triangle OBD we find that the individual amplitude A_n is given by

$$A_1=A_n=2r \sin \alpha \qquad \ldots(4)$$

Dividing (3) by (4), we get

$$\frac{A}{A_n}=\frac{2r \sin (\phi/2)}{2r \sin \alpha}=\frac{\sin N\alpha}{\sin \alpha} \qquad \ldots(5)$$

where use has been made of (2). Eliminating A_n between (1) and (5),

$$A=A_0\frac{\sin\beta}{\beta}\frac{\sin N\alpha}{\sin\alpha} \qquad \ldots(6)$$

$$\text{Intensity, } I=A^2=A_0{}^2\frac{\sin^2\beta}{\beta^2}\cdot\frac{\sin^2 N\alpha}{\sin^2\alpha}=I_0\frac{\sin^2 N\alpha}{\sin^2\alpha} \qquad \ldots(7)$$

where I_0 is the intensity for a single slit.

The principal maxima occur in directions for which the waves are all in phase.

$$\delta=2m\pi \text{ with } m=0, \pm 1, \pm 2, \ldots$$

$$\therefore \quad \alpha=\delta/2=m\pi \qquad \ldots(8)$$

We note that when the condition (8) is used in (7), the quotient ($\sin N\alpha/\sin\alpha$) becomes indeterminate. We, therefore, resort to L'Hospital's rule i.e., differentiate both the numerator and the denominator of (7) and use (8),

$$\lim_{\alpha\to m\pi}\frac{\sin N\alpha}{\sin\alpha}=\lim_{\alpha\to m\pi}\frac{N\cos N\alpha}{\cos\alpha}=\pm N$$

Then (7) becomes

$$I_{max}=N^2 I_0$$

$$I_0/I_{max}=1/N^2$$

These secondary maxima bear a strong resemblance to those of the secondary maxima in the single slit pattern. Therefore, replacing I with I_k, we have

$$I_k/I_{max}=1/N^2$$

This establishes the result required in part (*b*) of the Problem.

Secondary maxima are positioned for the approximate value

$$\alpha=\frac{(2k+1)\pi}{2N}$$

In this case $|\sin N\alpha|=1$. Also, since N is very large α will be small and $\sin^2\alpha\simeq\alpha^2$. Also, $\sin\beta/\beta\simeq 1$, as β will also be small. From (7)

$$I_k=\frac{A_0{}^2}{\alpha^2}=\frac{A_0{}^2N^2}{(k+\frac{1}{2})^2\pi^2}=\frac{I_{max}}{(k+\frac{1}{2})^2\pi^2}$$

$$\frac{I_k}{I_{max}}=\frac{1}{(k+\frac{1}{2})^2\pi^2}$$

This is the result for part (*a*) of the Problem.

(*c*) As the number of slits becomes large the polygon of vectors approaches the arc of a circle and the analogy with the pattern due to a single slit of width equal to that of grating is complete. The diagrams for the grating become identical with those for a single slit if $N\alpha$ is replaced by β.

S.45.2.

θ	$\sin\theta$	m	$\sin\theta/m$
17.6°	0.30237	1	0.30237
−17.6°	0.30237	1	0.30237
37.3°	0.60599	2	0.30300
−37.1°	0.60321	2	0.30161
65.2°	0.90778	3	0.30259
−65.2°	0.90778	3	0.30259

$$(\sin\theta/m)_{av}=0.30242$$

$$\lambda=d\sin\theta/m=(1.732\times10^{-4}\text{ cm})(0.30242)$$

$$=5238\times10^{-8}\text{ cm}=5.238\text{ A}$$

S.45.3. (*a*)

θ	$\sin\theta$	m	$\sin\theta/m$
6°40′	0.11609	1	0.11609
13°30′	0.23345	2	0.11672
20°20′	0.34749	3	0.11583
35°40′	0.58306	5	0.11661

$$(\sin\theta/m)_{av}=0.1163)$$

$$\lambda=d\sin\theta/m=(5.04\times10^{-4}\text{ cm})(0.1163)=5861\text{ A.}$$

(*b*) The missing fourth order must lie between $\theta=20°20'$ and $\theta=35°40'$.

With $\theta=20°20'$

$$a=\frac{5861\times10^{-8}\text{ cm}}{0.34749}=1.69\times10^{-4}\text{ cm}=1.69\text{ micron.}$$

With $\theta=35°40'$

$$a=\frac{5861\times10^{-8}\text{ cm}}{0.58306}=1.0\times10^{-4}\text{ cm}=1.0\text{ micron.}$$

Thus, the slit width must lie between 1.0 and 1.69 microns.

S.45.4. Grating equation is

$$d\sin\theta=m\lambda \qquad \ldots(1)$$

where d is the distance between the rulings and m, which is an integer, is the order of diffraction.

Diffrentiating (1) with respect to λ,

$$d \cos\theta \frac{d\theta}{d\lambda} = m$$

or $$\frac{d\theta}{d\lambda} = \frac{m}{d \cos \theta} = \frac{m}{d\sqrt{1-\sin^2 \theta}} = \frac{m}{\sqrt{d^2 - d^2 \sin^2 \theta}}$$

$$= \frac{m}{\sqrt{d^2-(m^2\lambda^2)}} = \frac{1}{\sqrt{(d^2/m^2)-\lambda^2}}$$

If the wavelengths are crowded then $d\lambda$ can be replaced by $\Delta\lambda$ and $d\theta$ by $\Delta\theta$.

Thus, $\Delta\theta = \Delta\lambda/\sqrt{(d^2/m^2)-\lambda^2}$.

S.45.5. (*a*) $d \sin \theta = m\lambda$ ...(1)

Set $\theta = 90°$

Then, $m = d/\lambda = \frac{9000 \text{ A}}{6000 \text{ A}} = 1.5$.

Only $m=1$ is allowed.

Thus, there can be only one line on each side of the central maximum.

(*b*) $d \sin \theta = m\lambda$

Set $m=1$

$$\sin \theta = \frac{\lambda}{d} = \frac{6000 \text{ A}}{9\ 00 \text{ A}} = 0.667 \qquad ...(2)$$

$\theta = 42°$

The angular width is

$$\Delta\theta = \frac{\lambda}{dN \cos \theta} = \frac{\tan \theta}{N}$$

$$= \frac{\tan 42°}{1000} = 9 \times 10^{-4} \text{ radians}$$

where use has been made of (2).

(*c*) Resolving power, $R = \lambda/\Delta\lambda$. ...(3)

Now, $\Delta\theta = \frac{m\,\Delta\lambda}{d \cos \theta}$ ...(4)

Multiply (3) and (4) to get

$$R\,\Delta\theta = \frac{m\lambda}{d \cos \theta} = \tan \theta \qquad ..(5)$$

where use has been made of (1).

$\therefore \quad \Delta\theta = \tan \theta/R$

S.45.6. (*a*) $R=\lambda/\Delta\lambda=Nm$ ...(1)

Now, $\nu=c/\lambda$ ...(2)

$$\therefore |\Delta\nu|=c\frac{\Delta\lambda}{\lambda^2}=\frac{c}{\lambda}\frac{\Delta\lambda}{\lambda}=\frac{c}{Nm\lambda} \qquad ...(3)$$

where use has been made of (1).

(*b*) Path difference between the extreme rays is

$$\delta=Nd\sin\theta. \qquad ...(4)$$

Therefore time of flight difference,

$$\Delta t=\frac{\delta}{c}=\frac{Nd\sin\theta}{c}. \qquad ...(5)$$

(*c*) Multiply (3) and (5)

$$(\Delta\nu)(\Delta t)=\frac{c}{Nm\lambda}\frac{Nd\sin\theta}{c}=\frac{d\sin\theta}{m\lambda}=1$$

where use has been made of the grating equation. $d\sin\theta=m\lambda$.

S.45.7. Refer to Textbook Fig. 45.14 (*a*) and 45.14 (*b*). Next five smaller interplaner spacings are shown in the sketch of Fig. S.45.7.

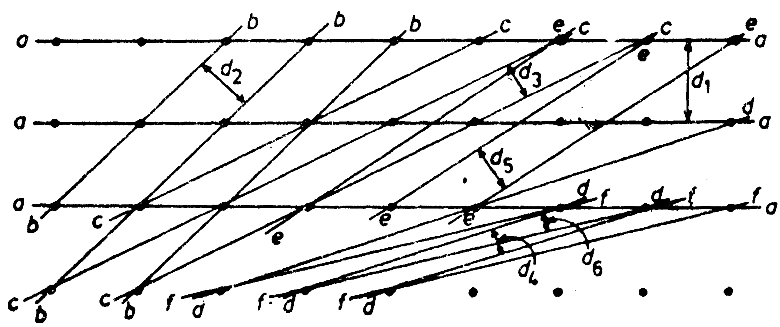


Fig. S.45.7

$$d_1=a_0$$

$$d_2=\frac{a_0}{\sqrt{2}}\ ;\ h=1,\ k=1$$

$$d_3=\frac{a_0}{\sqrt{5}}\ ;\ h=1,\ k=2$$

$$d_4=\frac{a_0}{\sqrt{10}}\ ;\ h=1,\ k=3$$

$$d_5=\frac{a_0}{\sqrt{13}}\ ;\ h=2,\ k=3$$

$$d_6=\frac{a_0}{\sqrt{17}}\ ;\ h=1,\ k=4,$$

46 POLARIZATION

46.1. Let I_0 be the intensity of the incident unpolarized light, I_1 the intensity of beam through the first sheet and I_2 the intensity transmitted through the second sheet.

By Problem, $I_1 = \frac{1}{2} I_0$

$$I_2 = I_1 \cos^2 \theta$$

(*a*) $$\frac{I_2}{I_1} = \cos^2 \theta = \frac{1}{3}$$

$\therefore$ $\cos \theta = \pm 1/\sqrt{3} = \pm 0.577$, or $\theta = \pm 55°$.

(*b*) $$I_2 = \frac{I_0}{3} = I_1 \cos^2 \theta = \frac{I_0}{2} \cos^2 \theta$$

$\therefore$ $\cos \theta = \pm \sqrt{\frac{2}{3}} = \pm 0.8165$ or $\theta = \pm 35°$.

46.2. Let I_0 be the incident intensity, I_1, I_2, I_3 and I_4 the intensity from successive sheets. Then the transmitted intensity of light

$$I_4 = I_3 \cos^2 \theta$$

Also, $I_3 = I_2 \cos^2 \theta$

$$I_2 = I_1 \cos^2 \theta$$

$$I_1 = \tfrac{1}{2} I_0$$

$\therefore$ $I_4 = I_3 \cos^2 \theta = I_2 \cos^4 \theta = I_1 \cos^6 \theta = (I_0/2) \cos^6 \theta$.

$\therefore$ Fraction of incident intensity that is transmitted is given by

$$\frac{I_4}{I_0} = \frac{1}{2} \cos^6 \theta = \frac{1}{2} \cos^6 30° = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^6 = \frac{27}{128}.$$

46.3. (*a*) $E_x = E \sin (kz - \omega t)$ ...(1)

$E_y = E \cos (kz - \omega t)$ ...(2)

Square (1) and (2) and add

$E_x^2 + E_y^2 = E^2$ ...(3)

This is an equation of a circle. It is circularly polarized. The wave has the form

$$\mathbf{E} = \mathbf{i} E_x + \mathbf{j} E_y = E[\mathbf{i} \sin (kz - \omega t) + \mathbf{j} \cos (kz - \omega t)]$$

We can examine its behaviour at some fixed point in space, say $z=0$. At $t = 0$, $T/4$, $T/2$, $3T/4$ and T, $\mathbf{E}(0, t)$ has values of $E\mathbf{j}$, $-E\mathbf{i}$, $-E\mathbf{j}$, $E\mathbf{i}$ and $E\mathbf{j}$, respectively. These values are indicated in Fig 46.3 (*a*) over one period, using a right-handed system. $\mathbf{E}$ has

constant magnitude but rotates counter-clockwise (looking toward the source). The field is therefore left circularly polarised.

(*b*) $E_x = E\cos(kz-\omega t)$ ...(4)

$E_y = E\cos(kz-\omega t+\pi/4)$ ...(5)

Equations (4) and (5) can be combined to obtain

$E_x^2+E_y^2-\sqrt{2}\,E_xE_y=E^2/2.$ ...(6)

This is an equation of an ellipse whose major axis is tilted at an angle 45° to the E_x axis.

We may write (4) and (5) in the vector form

$$\mathbf{E}(z,t)=\mathbf{i}E\cos(kz-\omega t)+\mathbf{j}E\cos(kz-\omega t+\pi/4).$$

We can examine its behaviour at $z=0$ and for various values of t.

$$\mathbf{E}(0,0)=\mathbf{i}E+\mathbf{j}\frac{E}{\sqrt{2}}$$

$$\mathbf{E}\left(0,\frac{T}{4}\right)=0+\mathbf{j}\frac{E}{\sqrt{2}}$$

$$\mathbf{E}\left(0,\frac{T}{2}\right)=-\mathbf{i}E-\mathbf{j}\frac{E}{\sqrt{2}}$$

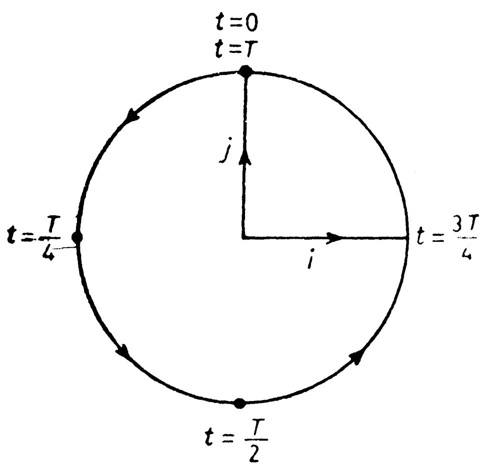


Fig. 46.3 (a)

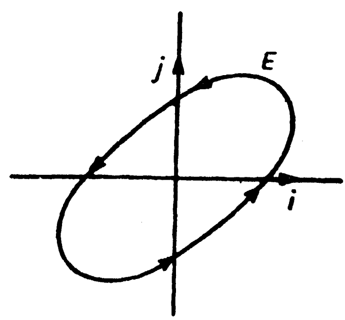


Fig. 46.3 (b)

$$\mathbf{E}\left(0,\frac{3T}{4}\right)=0-\mathbf{j}\frac{E}{\sqrt{2}}$$

$$\mathbf{E}(0,T)=\mathbf{i}E+\mathbf{j}\frac{E}{\sqrt{2}}$$

It is seen from Fig. 46.3 (*b*), that the *E*-field rotates counter-clockwise and is therefore left handed.

(*c*) $E_x = E \sin(kz - \omega t)$

$E_y = -E \sin(kz - \omega t)$

$\therefore \quad E_y = -E_x$

This is an equation of a straight line. The light is plane polarized with the major axis inclined at an angle of 135° with the E_x axis.

46.4. (*a*) For water $\tan \theta_p = n = 1.33$

$\therefore \quad \theta_p = 53°$

(*b*) Yes, the angle does depend upon the wavelength as *n* is a function of λ.

46.5. From Textbook Fig. 41.2, we note that the refractive indices range from 1 470 to 1.455 for the white light. The corresponding polarizing angles are

$$\theta_p = \tan^{-1} 1.470 = 55°46'$$

and
$$\tan^{-1} 1.455 = 55°30'$$

46.6. (*a*) For the ordinary ray,

$$\sin r_0 = \frac{\sin i}{n_0} = \frac{\sin 45°}{1.658} = 0.4265$$

$\therefore \quad r_0 = 25°14'$

For the extraordinary ray

$$\sin r_e = \frac{\sin i}{n_e} = \frac{\sin 45°}{1.486} = 0.4758$$

$\therefore \quad r_e = 28°25'$

$$\tan r_0 = \tan 25°14' = 0.471 = FC/BF$$

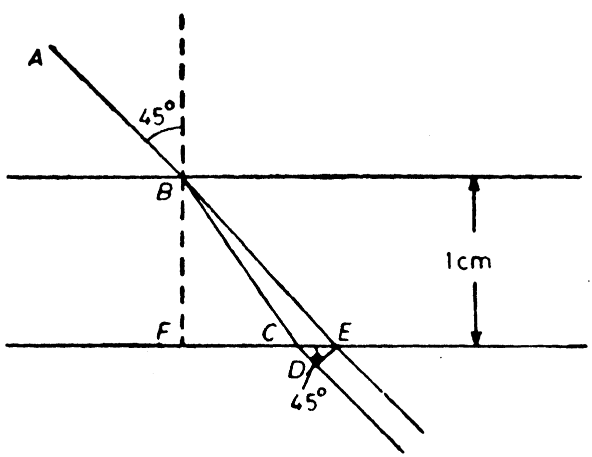


Fig. 46.6

Since $BF=1$ cm,

$FC=0.471$ cm

$\tan r_e=\tan 28°25'=0.541=FE/BF$

$\therefore$ $FE=0.541$ cm

$\therefore$ $CE=FE-FC=0.541 \text{ cm}-0.471 \text{ cm}=0.07$ cm

$DE=CE/\sqrt{2}=0.05 \text{ cm}=0.5$ mm

(*b*) Ray x is extraordinary whilst ray y is ordinary.

(*c*) **E** in ray y lies in the plane of figure whilst **E** in ray x lies at right angles to the plane of the figure.

(*d*) If a polarizer is placed in the incident beam and rotated, for every rotation of 90° one or the other ray will be alternately extinguished.

46.7. Set the prism in the minimum deviation position for the ordinary ray and then for extra ordinary ray and if the angles of minimum deviation D_0 and D_e, respectively for the two rays be determined then the refractive indices can be found out using the following formulae.

$$n_0=\sin \tfrac{1}{2}(A+D_0)/\sin \tfrac{1}{2} A$$

$$n_e=\sin \tfrac{1}{2}(A+D_e)/\sin \tfrac{1}{2} A$$

with $A=60°$.

46.8. Thickness of the quarter-wave plate,

$$t=\frac{\lambda}{4\,(n_e-n_0)}=\frac{5890\times 10^{-8}\text{ cm}}{4\,(1.6117-1.6049)}$$

$$=0.00217 \text{ cm} = 0.0217 \text{ mm}.$$

46.9. Let the two linearly polarized plane waves be given by

$$\mathbf{E}_1(\mathbf{r}, t)=\mathbf{E}_{01}\cos(\mathbf{k}_1.\mathbf{r}-\omega t+\phi_1)$$

$$\mathbf{E}_2(\mathbf{r}, t)=\mathbf{E}_{02}\cos(\mathbf{k}_2.\mathbf{r}-\omega t+\phi_2)$$

Then the resultant field is

$$\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2$$

The intensity is

$$I=\langle \mathbf{E}^2 \rangle$$

where $E^2=\mathbf{E}.\mathbf{E}$

i.e. $E^2=(\mathbf{E}_1+\mathbf{E}_2).(\mathbf{E}_1+\mathbf{E}_2)=E_1^2+E_2^2+2\mathbf{E}_1.\mathbf{E}_2$

Taking the average we find

$$I=I_1+I_2+I_{12}$$

where $I_1=\langle E_1^2\rangle$, $I_2=\langle E_2^2\rangle$ and $I_{12}=2\langle \mathbf{E}_1.\mathbf{E}_2\rangle$,

the last being the inteference term. But since $\mathbf{E}_1$ and $\mathbf{E}_2$ are perpendicular to each other, the dot product $\mathbf{E}_1.\mathbf{E}_2$ vanishes and consequently

$$I=I_1+I_2$$
$$=<E_1^2>+<E_2^2>$$

$$<E_1^2>=\frac{E_{01}^2}{T}\int \cos^2(\mathbf{k}_1.\mathbf{r}-\omega t+\phi_1)\,dt=\frac{E_{01}^2}{2}$$

Similarly $<E_2^2>=\dfrac{E_{02}^2}{2}$

Thus $\quad I=\dfrac{E_{01}^2}{2}+\dfrac{E_{02}^2}{2}$

which is independent of the phase difference. Hence, interference effects cannot be produced since the intensity remains constant as one moves from point to point in space.

46.10. Consider a parallel beam of circularly polarized light of cross-section A to fill up a box of length d. If U is the energy and ω the angular frequency then the angular momentum is by Textbook Eq. 46.5 given by

$$L=\frac{U}{\omega}$$

But $\quad L_v=\dfrac{L}{V}=\dfrac{U}{\omega V}=\dfrac{U}{\omega dA}$

where we have written $V=dA$

Also, the distance travelled $d=ct$ so that

$$L_v=\frac{U}{\omega ctA}=\frac{P}{\omega c}$$

since $\quad P=U/tA.$

46.11. Rate of transfer of angular momentum,

$$=\frac{P}{\omega}=\frac{P}{2\pi\nu}=\frac{P\lambda}{2\pi c}$$

$$=\frac{(100\text{ watts})(5\times10^{-7}\text{ meter})}{(2\pi)(3\times10^8\text{ meter/sec})}=2.65\times10^{-14}\text{ kg-m}^2/\text{sec}^2.$$

If the angular momentum is transferred in time t secs, then

$$2.65\times10^{-14}\,t=L'=I'\omega'$$

where L' is the angular momentum acquired by the flat disk. $I'=\frac{1}{2}MR^2$ is the rotational inertia about its axis and ω' is the angular frequency of rotation.

$$t = \frac{\frac{1}{2}(1.0^{-6}\ \text{kg})(2.5\times 10^{-8}\ \text{meter})^2(2\pi\times 1.0\ \text{rev/sec})}{2.65\times 10^{-14}\ \frac{\text{kg}-m^2}{\text{sec}^2}}$$

$$=7400\ \text{sec}=2.06\ \text{hrs}.$$

SUPPLEMENTARY PROBLEMS

S.46.1. Let I_p be the original intensity of plane polarized light and I_0 that of randomly polarised light. In the light transmitted through the polaroid sheet for the randomly polarized component

$$I_o' = \tfrac{1}{2}\, I_0 \qquad \ldots(1)$$

and for plane polarized component,

$$I_p' = I_p \cos^2\theta \qquad \ldots(2)$$

where θ is the angle between the plane of polarization and the characteristic direction of the polaroid. The maximum transmission intensity is given by adding (1) and (2) and setting $\theta=0$.

$$I'_{(max)} = I_p' + I_o' = I_p + \tfrac{1}{2}\, I_o \qquad \ldots(3)$$

and the minimum transmission intensity is given by setting $\theta=90^0$.

$$I'_{(min)} = I_p' + I_o' = 0 + \tfrac{1}{2}\, I_o \qquad \ldots(4)$$

Dividing (3) by (4)

$$\frac{I'_{(max)}}{I'_{(min)}} = \frac{I_p + \frac{1}{2} I_o}{\frac{1}{2} I_o} = \frac{5}{1}$$

whence $I_p = 2I_o$

Thus, the relative intensity of the polarized component in the incident light is

$$\frac{I_p}{I_p+I_o} = \frac{2I_o}{2I_o+I_o} = \frac{2}{3}$$

and that of randomly polarized light is

$$\frac{I_o}{I_p+I_o} = \frac{I_o}{2I_o+I_0} = \frac{1}{3}$$

S.46.2. (*a*) Let the light pass through a stack of polaroid sheets such that each sheet causes the plane of polarization to rotate through an angle less than 90° but the total angle of rotation addes up to 90°.

(*b*) Transmitted intensity through the first polaroid sheet

$$I_1 = I_o \cos^2\theta$$

where I_o is the original intensity. Similarly, the transmitted intensity through the second sheet will be

$$I_2=I_1 \cos^2 \theta=I_o \cos^4 \theta$$

If N sheets of polaroid are used, then

$$I_N=I_o(\cos \theta)^{2N}$$

where we have assumed that angle of rotation through each sheet is the same.

By Problem,

$$I_N/I_o=0.95$$

Furthermore, $\theta=\frac{90^0}{N}=\frac{\pi}{2N}$ radians

$$\therefore \left(\cos \frac{\pi}{2N}\right)^{2N}=0.95$$

As N is expected to be large, the cosine function can be approximated by the power series retaining only the first two terms and then use the binomial expansion as follows:

$$\left(\cos \frac{\pi}{2N}\right)^{2N}=\left(1-\frac{\pi^2}{8N^2}+\dots\right)^{2N}\simeq\left[1-\frac{2N\,\pi^2}{8N^2}+\dots\right]=0.95$$

Solving for N, we find

$$N=5\pi^2=50.$$

46.3. The function of a polaroid is to convert unpolarised light into plane polarised light. The quarter wave plate introduces a path difference of $\lambda/4$ or a phase difference of $\pi/2$ between the two emergent waves, where λ is the wavelength of the incident monochromatic light. Its function is to convert in general the incident plane-polarised light into elliptically-polarised light though the special cases of emergent plane-polarised light occur for $\theta=0^\circ$ or 90° and circularly polarised light for $\theta=45^\circ$, where $90^\circ-\theta$ is the angle between the electric vector in the incident plane polarised light and the optic axis. Upon reversing the direction of the light the role of the quarter-wave plate is to change the elliptically-polarised light to plane-polarised light.

Let the side A have the polaroid and let the polaroid axis be oriented at 45° to the principal axes of the quarter-wave plate. The unpolarised light first enters the polaroid and the emergent plane-polarised light upon traversing through the quarter wave plate is changed to circularly-polarised light which after reflection from the coin returns and on traversing the quarter-wave plate is changed back to the plane-polarised light but now the polaroid is "crossed" and so the extinction of light occurs. Thus, the same combination of polaroid and the quarter-wave plate acts both as polariser and

analyser. On the other hand if this combination is placed with the quarter-wave plate away from the coin (with the face *A* against the coin) then unpolarised light would first enter the quarter-wave plate and will be unaffected, and the resultant light upon traversing the polaroid would not be extinguished.

S.46.4. For the right circularly-polarised light the eletric vector **E** rotates clockwise on a circle around the direction of propagation as we look toward the source.

(*a*) Upon reflection the direction of propagation is reversed and the reflected beam becomes left circularly-polarized.

(*b*) As the direction of light beam is reversed, the direction of linear momentum of light is also changed.

(*c*) Angular momentum is a pseudo vector (axial vector) i.e. a vector given by the vector product of two polar vectors. Under mirror reflection, the direction of angular momentum of light does not change. As the incident light is right circularly-polarized, its spin points in the direction of propagation. Upon reflection the light has become left circularly-polarized and its direction of propagation has been reversed so that its spin still points in the original direction. In other words the direction of spin has not changed.

(*d*) The impact of light beam on the mirror causes radiation pressure.

47 LIGHT AND QUANTUM PHYSICS

47.1. Planck's formula is

$$R_\lambda = \frac{C_1}{\lambda^5}\left(\frac{1}{e^{C_2/\lambda T}-1}\right)$$

with $C_1 = 2\pi c^2 h$ and $C_2 = hc/k$

The peak in the intensity distribution at the given temperature T is obtained from the condition

$$\frac{\partial R_\lambda}{\partial \lambda} = 0$$

This leads to $\lambda_m = \dfrac{C^2}{5T}\left(\dfrac{1}{1-e^{-C_2/T\lambda_m}}\right)$

This is a transcendental equation in λ_m.

Set $C_2/T\lambda_m = x$

then, $\dfrac{x}{1-e^{-x}} = 5.$

The equation is satisfied to a good approximation with $x=5$.

$$\therefore \quad \frac{C^2}{T\lambda_m} = 5$$

or $$\frac{hc}{kT\lambda_m} = 5$$

or $$\lambda_m = \frac{hc}{5kT}$$

$$\therefore \quad \lambda_m = \frac{(6.63\times10^{-34}\text{ joule-sec})(3\times10^8\text{ meter/sec})}{(5)(1.38\times10^{-23}\text{ joule/°K})(6000\text{ °K})}$$
$$= 4.8\times10^{-7}\text{ meter} = 4800\text{ A.}$$

47.2. Area under the lower curve (Tungsten) is 2.35 cm^2. Area under the upper curve (Cavity radiator) is about 9 cm^2

$\therefore$ Emissivity of tungsten $= 2.35/9 = 0.26$

47.3. $T = 6000$ °K

Area of the hole $= \pi/4\ (0.01)^2$ cm^2 $= 7.85\times10^{-5}$ cm^2

$$\text{Mean } \lambda = \frac{5500+5510}{2} = 5505 \text{ A} = 5.505 \times 10^{-5} \text{ cm}$$

$$d\lambda = 10 \text{ A} = 10^{-7} \text{ cm.}$$

$$R_\lambda d\lambda = \frac{2\pi c^2 h d\lambda}{\lambda^5\left(e^{hc/\lambda kT} - 1\right)}$$

Let N photons/sec energy through the hole. Power radiated through the hole $= Nh\nu$

$$\therefore\ AR_\lambda d\lambda = Nh\nu, \text{ or } N = \frac{AR_\lambda d\lambda}{h\nu}$$

or
$$N = \frac{2\pi c d\lambda A}{\lambda^4\left(e^{hc/\lambda kT} - 1\right)}$$

$$\text{Now, } \frac{hc}{\lambda kT} = \frac{6.63 \times 10^{-34} \text{ joules-sec})(3 \times 10^8 \text{ meter/sec})}{(5.505 \times 10^{-7} \text{ m})(1.38 \times 10^{-23} \text{ joule/°K})(6000\text{°K})}$$

$$= 4.36$$

$$\therefore\ N = \frac{(2\pi)(3 \times 10^{10} \text{ cm/sec})(10^{-7} \text{ cm})(7.85 \times 10^{-5} \text{ cm}^2)}{(5.505 \times 10^{-5} \text{ cm})^4 (e^{4.36} - 1)}$$

$$= 2.1 \times 10^{15}.$$

47.4. $T = 4000\text{°K}$

Area, $A = \pi(0.0025)^2 = 1.94 \times 10^{-5}$ meter^2

$$\text{Mean } \lambda = \frac{0.4+0.7}{2} \mu m = 0.55\ \mu m = 5.5 \times 10^{-7} \text{ meter}$$

$$d\lambda = 0.7 - 0.4 = 0.3 \times 10^{-4} \text{ cm} = 3 \times 10^{-5} \text{ cm} = 3 \times 10^{-7} \text{ meter}$$

(*a*) Energy/sec escaping from the hole in the visible region

$$AR_\lambda d\lambda = \frac{2\pi c^2 h d\lambda A}{\lambda^5\left(e^{hc/\lambda kT} - 1\right)}$$

$$\text{Now, } \frac{hc}{\lambda kT} = \frac{(6.63 \times 10^{-34} \text{ joules-sec})(3 \times 10^8 \text{ meter/sec})}{(5.5 \times 10^{-7} \text{ meter})(1.38 \times 10^{-23} \text{ joule/°K})(4000\text{°K})}$$

$$= 6.55$$

$$\therefore\ AR_\lambda d\lambda$$

$$= \frac{(2\pi)(3 \times 10^8 \text{ meter/sec})^2 (6.63 \times 10^{-34} \text{ j-s})(3 \times 10^{-7} \text{ m})(1.94 \times 10^{-5} \text{ m}^2)}{(5.5 \times 10^{-7} \text{ meter})^5 (e^{6.55} - 1)}$$

$= 63$ joules/sec $= 63$ watts

(b) Total cavity radiation, $R_c=\sigma T^5$

where σ is Stefen-Boltzmann constant.

$$\sigma=5.67\times10^{-8}\text{ watt/(meter}^2)(^\circ\text{K}^4)$$

Total cavitation radiation escaping through the hole

$$=R_cA=(5.67\times10^{-8})\times(4000)^4\times(1.94\times10^{-5})=285\text{ watts.}$$

$\therefore$ Fraction of enei gy in the visible region$=\frac{63}{285}=0.22$

47.5. Planck's law is, $R_\lambda=\frac{C_1}{\lambda^5}\frac{1}{\left(e^{C_2/\lambda T}-1\right)}$

If λ is small or T is small then the exponential term in the paranthesis is much larger compared to unity.

$$R_\lambda\simeq\frac{C_1}{\lambda^5 e^{C_2/\lambda T}}\text{ (Wien's law)}$$

47.6. Solar radiation falls on earth at the rate of 2.0 cal/cm²-min, or $2.0\times2.613\times80^{19}$ ev/cm²-min.

Energy of each photon of $\lambda=5500\text{A}^\circ$ is

$$E=h\nu=\frac{hc}{\lambda}$$

$$=(4.14\times10^{-15}\text{ ev-sec})(3\times10^{10}\text{ cm/sec})/(5.5\times10^{-5}\text{ cm})$$

$$=2.26\text{ ev}$$

Therefore, number of photons incident/cm²-min

$$=\frac{2.0\times2.613\times10^{19}\text{ ev/cm}^2\text{-min}}{(2.26\text{ ev})}=2.31\times10^{19}$$

47.7. $E=h\nu=\frac{hc}{\lambda}=\frac{(4.14\times10^{-15}\text{ ev-sec})(3\times10^{10}\text{ cm/sec}}{(21\text{ cm})}$

$$=\frac{(4.14\times10^{-15}\text{ ev-sec})(3\times10\text{ cm/sec})}{(21\text{ cm})}=5.9\times10^{-6}\text{ ev.}$$

47.8. For orange light, $E=\frac{hc}{\lambda}=\frac{(4.14\times10^{-15}\text{ ev-sec})(3\times10^{10}\text{ cm/sec})}{(6800\times10^{-8}\text{ cm})}$

$$=1.826\text{ ev}$$

Sodium cannot show photoelectric effect as the photon energy is less than 2.3 ev, the energy required to remove an electron from sodium.

47.9. (*a*) Energy of photon$=\dfrac{hc}{\lambda}$

$$=\frac{(4.14\times10^{-15}\text{ ev-sec})(3\times10^{10}\text{ cm/sec})}{(2000\times10^{-8}\text{ cm})}=6.21$$

Energy of fast photoelectrons$=h\nu-W=6.2-4.2=2.0$ ev.

(*b*) Energy of slowest photo electrons is zero since originally electrons in the metals have Fermi-distribution and for these electrons which are sitting down the well, greater energy is required. Therefore, the photo-electrons have an energy spectrum, the minimum energy being zero.

(*c*) Stopping potential is 2 volts.

(*d*) Cut-off wavelength corresponding to threshold energy of 4.2 ev is

$$\lambda=\frac{hc}{E}\frac{(4.14\times10^{-15}\text{ ev-sec})(3\times10^{10}\text{ cm/sec})}{4.2\text{ ev}}$$

$$\simeq 3\times10^{-5}\text{ cm}=3000\text{A}^\circ$$

47.10. Photo-electric effect equation is $h\nu-E_0=V_0$ with $E_0=2.3$ ev for Lithium. For a given value of V_0, the frequency of the incident radiation is calculated from $\nu=\dfrac{E_0+V_0}{h}$. The calculated values of ν are tabulated. Fig. 47.10 shows the plot of stopping potential *versus* the frequency of the incident radiation.

V_0	$h\nu$	ν (Cps)
0	2.3 ev	5.5×10^{14}
0.2	2.5 ev	6.0×10^{14}
0.4	2.7 ev	6.5×10^{14}
0.7	3.0 ev	7.2×10^{14}
1.0	3.3 ev	8.0×10^{14}
1.5	3.8 ev	9.2×10^{14}

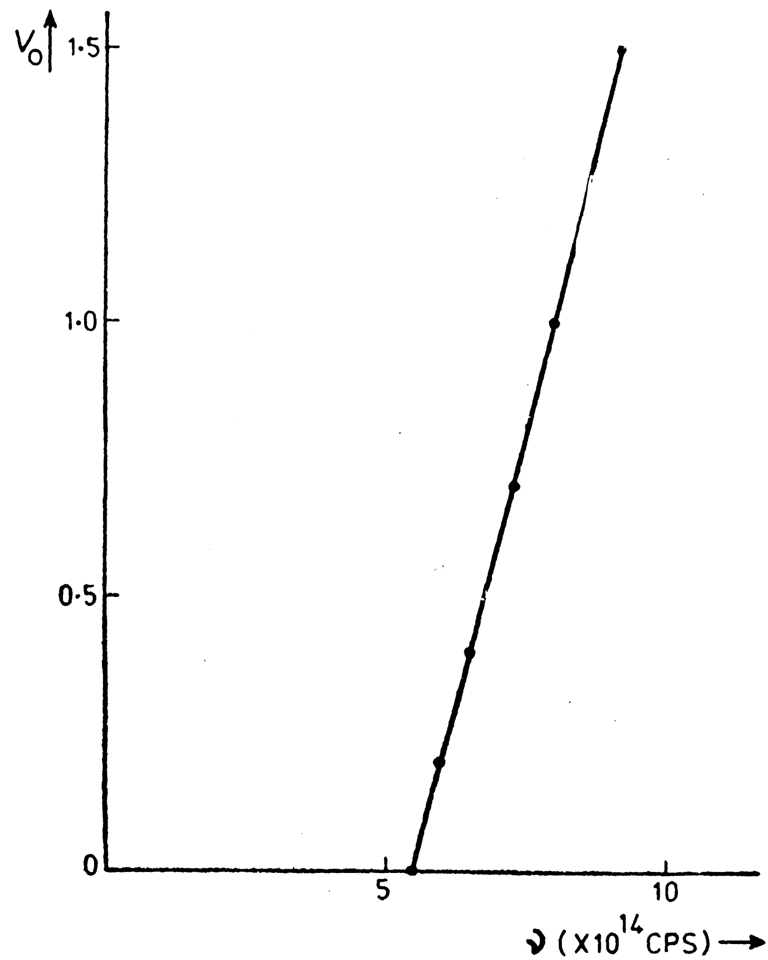


Fig. 47.10

47.11. Intensity of light source $=10^{-5}$ watts.

Binding energy of electron $=20$ ev $=20\times1.6\times10^{-19}$ joules
$=3.2\times10^{-18}$ joules

Target area $=\pi\,(10^{-10}\text{ meter})^2=10^{-20}\,\pi\text{ meter}^2$

The area of a 5-meter sphere centered on the light source is
$4\pi\,(5\text{ meter})^2=100\pi\text{ meter}^2$

If the light source radiates uniformly in all directions, the rate P at which energy falls on the target is given by

$$P=(10^{-5}\text{ watts})\frac{(10^{-20}\,\pi\text{ meter}^2)}{(100\,\pi\text{ meter}^2)}=10^{-27}\text{ joules/sec.}$$

Assuming that all this power is absorbed, the time required for the ejection of electron is

$$t=\frac{(3.2\times10^{-18}\text{ joules})}{(10^{-27}\text{ joules/sec})}=3.2\times10^{9}\text{ secs}\simeq 100\text{ years}$$

47.12. $\lambda=5890\times10^{-10}$ m.

(*a*) Let N photons be emitted/sec.

$$100\text{ watts}=h\nu N=\frac{hcN}{\lambda}$$

$$\text{or } N=\frac{100\,\lambda}{hc}=\frac{100\times5890\times10^{-10}}{6.63\times10^{-34}\times3\times10^{8}}=2.96\times10^{20}\text{ per sec}$$

At distance r=cm, number of photons going through 1 cm^2 is $\frac{N}{4\pi r^2}$.

At distance r cm, density is $\frac{N}{4\pi cr^2}=10$

$$\therefore \qquad r=\sqrt{\frac{N}{40\pi c}}=\sqrt{\frac{2.96\times10^{20}}{40\times3\times10^{10}\pi}}=8863\text{ cm}=88.6\text{ meter.}$$

(*b*) As the density is inversely proportional to r^2, the density at 2.0 meters from the lamp$=(10)\left(\frac{88.6}{2}\right)^2=1.96\times10^4\ /\text{cm}^3$.

47.13. Suppose a photon of energy $h\nu$ is completely absorbed by a free electron. Then the photo-electron must be ejected in the forward direction in order to conserve momentum. Conservation of energy gives

$$h\nu+m_0c^2=\sqrt{c^2p^2+m_0^2c^4} \qquad \ldots(1)$$

Conservation of momentum gives

$$\frac{h\nu}{c}=p \qquad \ldots(2)$$

Using (2) in (1),

$$h\nu+m_0c^2=\sqrt{h^2\nu^2+m_0^2c^4} \qquad \ldots(3)$$

Squaring both sides and simplifying

$2h\nu m_0c^2=0$, which is absurd since

$h\nu\neq0$ and $m_0c^2\neq0$.

47.14. $\frac{\Delta E}{E}=1\Big/\left(\frac{1}{1+2\alpha\sin^2\theta/2}\right)$, where $\alpha=\frac{h\nu}{m_0c^2}$.

for $\theta=90°$

$$\frac{\Delta E}{E}=1\Big/\left(1+\frac{1}{\alpha}\right)=1\Big/\left(1+\frac{m_0c^2}{h\nu}\right)$$

(*a*) for $\lambda=3.0$ cm, $h\nu=h\frac{c}{\lambda}$

$$=(4.14\times10^{-15}\text{ ev-sec})\frac{(3\times10^{10}\text{ cm/sec})}{3\text{ cm}}=4.1\times10^{-5}\text{ ev.}$$

$$\frac{\Delta E}{E}=1\Big/\left(1+\frac{0.51\times10^{6}\text{ ev}}{4.1\times10^{-5}\text{ ev}}\right)\simeq8\times10^{-11}.$$

Percentage change in energy $=(\Delta E/E)\ 100=8\times10^{-11}\times100$
$=8\times10^{-9}\ \%$

which is practically zero.

(*b*) For $\lambda=5000\text{A}^\circ$; $h\nu=\frac{hc}{\lambda}=\frac{(4.14\times10^{-15}\text{ ev-sec})(3\times10^{10}\text{ cm/sec})}{5000\times10^{-8}\text{ cm})}$

$=2.48$ ev.

$$\frac{\Delta E}{E}=1\Big/\left(1+\frac{0.51\times10^{6}\text{ ev})}{2.48\text{ ev}}\right)\simeq5\times10^{-6}$$

Percentage change in energy $=5\times10^{-6}\times100=5\times11^{-4}\ \%$

(*c*) For $\lambda=1.0\text{A}^\circ$; $h\nu=\frac{hc}{\lambda}=\frac{(4.14\times10^{-15}\text{ ev-sec})(3\times10^{10}\text{ cm/sec}}{(10^{-8}\text{ cm})}$

$=12420$ ev.

$$\frac{\Delta E}{E}=1\Big/\left(1+\frac{0.51\times10^{6}\text{ ev}}{12420\text{ ev}}\right)=\frac{1}{41.2}=0.0242$$

Percentage change in energy $=0.0242\times100=2.4\ \%$

(*d*) For $h\nu=1$ Mev

$$\frac{\Delta E}{E}=1\Big/\left(1+\frac{0.51}{1}\right)=\frac{1}{1.51}=0.66$$

$\therefore$ Percentage change in energy $=66\%$.

For small photon energy (small compared to rest-mass of electron i.e. 0 51 Mev) there is practically no loss of energy in the scattering process. But for high energies such as those associated with X-rays, energy loss may be significant.

47 15. $\frac{hc}{\lambda}=\frac{hc}{\lambda'}+m_0c^2\left[\frac{1}{\sqrt{1-v^2/c^2}}-1\right]$, Textbook (47-13)

$$\frac{m_0v}{\sqrt{1-\frac{v^2}{c^2}}}\cos\theta=\frac{h}{\lambda}-\frac{h}{\lambda'}\cos\phi,\text{ Textbook (47.15)}$$

$$\frac{m_0 v \sin\theta}{\sqrt{1-v^2/c^2}}=\frac{h}{\lambda'}\sin\phi, \text{ Textbook (47.16)}$$

Eliminating θ by the squaring (47.15) and (47.16) and adding,

$$\frac{m_0{}^2v^2}{1-v^2/c^2}=h^2\left(\frac{1}{\lambda^2}+\frac{1}{\lambda'^2}-\frac{2\cos\phi}{\lambda\lambda'}\right)$$

Cancelling the common term c in (47.13) and re-writing

$$\frac{m_0c}{\sqrt{1-v^2/c^2}}=\frac{h}{\lambda}-\frac{h}{\lambda'}+m^0c$$

Square the last equation and from the resultant equation substract the previous equation

$$\frac{m_0{}^2(c^2-v^2)}{1-v^2/c^2}=m_0{}^2c^2-\frac{2h^2}{\lambda\lambda'}+2m_0c\left(\frac{h}{c}-\frac{h}{\lambda'}\right)+\frac{2h^2\cos\phi}{\lambda\lambda'}$$

or $$m_0{}^2c^2=m_0c^2-\frac{2h^2}{\lambda\lambda'}(1-\cos\phi)+2m_0hc\,(1/\lambda-1/\lambda')$$

Cancelling m_0c^2 on both sides

$$2m_0hc\left(\frac{1}{\lambda}-\frac{1}{\lambda'}\right)=\frac{2h^2}{\lambda\lambda'}(1-\cos\phi)$$

Multiply throughout by $\lambda\lambda'$, and simplify,

$$\Delta\lambda=\lambda'-\lambda=\frac{h}{m_0c}(1-\cos\phi), \text{ Textbook (47.17).}$$

47.16. $\nu=\dfrac{2\pi^2me^4}{h^3}\left(\dfrac{1}{j^2}-\dfrac{1}{k^2}\right)$

First three longest wavelengths in the Balmer series are obtained by putting $j=2$ and $k=3$, 4 and 5.

$$\frac{2\pi^2me^4}{h^3}=(2\pi^2)\frac{(9.11\times10^{-28}\text{ gm})(4.8\times10^{-10}\text{ esu})^4}{(6.625\times10^{-27}\text{ erg-sec})^3}=3.28\times10^{15}$$

$\therefore$ $$\nu=3.28\times10^{15}\left(\frac{1}{j^2}-\frac{1}{k^2}\right)$$

(*a*) $$\nu_1=3.28\times10^{15}\left(\frac{1}{2^2}-\frac{1}{3^2}\right)=4.556\times10^{14}.$$

$\therefore$ $$\lambda_1=\frac{c}{\nu_1}=\frac{3\times10^{10}}{4.556\times10^{14}}=0.6585\times10^{-4}\text{ cm}=6585\text{ A}^\circ$$

$$\nu_2=3.28\times10^{15}\left(\frac{1}{2^2}-\frac{1}{4^2}\right)=0.6150\times10^{15}.$$

$$\lambda_2=\frac{3\times10^{10}}{0.6150\times10^{15}}=4908\times10^{-8}\text{cm}=4908\text{ A}^\circ.$$

$$\nu_3 = 3.28 \times 10^{15}\left(\frac{1}{2^2} - \frac{1}{5^2}\right) = 0.6888 \times 10^{15}$$

$$\lambda_3 = \frac{3 \times 10^{10}}{0.6888 \times 10^{15}} = 4.369 \times 10^{-5}\text{cm} = 4369\ \text{A}^\circ.$$

(*b*) $$\nu_\infty = 3.28 \times 10^{15}\left(\frac{1}{2^2} - \frac{1}{\infty}\right) = 0.82 \times 10^{15}$$

$$\lambda_\infty = \frac{3 \times 10^{10}}{0.82 \times 10^{15}} = 3.658 \times 10^{-5}\ \text{cm} = 3658\text{A}^\circ.$$

Balmer series lies between λ_1 and λ_∞ i.e. 6585A° and 3658A°

47.17. (*a*) $n = 1$

(*b*) $r = \dfrac{h^2}{4\pi^2 me^2}$ where e is in *e.s.u.*, m in gms and h in ergs-sec

$$= \frac{(6.63 \times 10^{-27})^2}{4\pi^2 \times 9.1 \times 10^{-28} \times (4.8 \times 10^{-10})^2} = 0.53 \times 10^{-8}\ \text{cm}$$

(*c*) $L = \dfrac{h}{2\pi} = 6.63 \times 10^{-27}/2\pi = 1.1 \times 10^{-27}$ erg-sec

(*d*) $p = mv = \dfrac{2\pi me^2}{h} = \dfrac{mc^2}{c}\dfrac{e^2}{c}\dfrac{e^2}{\hbar c} = \dfrac{0.511 \times 10^6}{137}\dfrac{1}{c}$

$= 3730$ ev/c.

(*e*) $\omega = \dfrac{v}{r} = \dfrac{8\pi^3 me^4}{h^3} = \dfrac{(8\pi^3)(9.11 \times 10^{-28}\ \text{gm})(4.8 \times 10^{-10}\ \text{esu})^4}{(6.63 \times 10^{-27}\ \text{erg-sec})^3}$

$= 4.1 \times 10^{16}$ radians/sec.

(*f*) $v = \dfrac{2\pi e^2}{h} = \dfrac{(2\pi)(4.8 \times 10^{-10}\ esu)^2}{(6.63 \times 10^{-27}\ \text{erg-sec})} = 2.18 \times 10^8$ cm/sec.

(*g*) $F = \dfrac{e^2}{r^2} = \dfrac{16\pi^4 m^2 e^6}{h^4} = \dfrac{(16\pi^4)(9.11 \times 10^{-28}\ \text{gm})^2(4.8 \times 10^{-10}\ esu)^6}{(6.63 \times 10^{-27}\ \text{erg-sec})^4}$

$= 8.17 \times 10^{-3}$ dynes.

(*h*) $a = \dfrac{16\pi^4 me^6}{h^4} = \dfrac{8.17 \times 10^{-3}\ \text{dyne}}{9.1 \times 10^{-28}\ \text{gm}} = 9 \times 10^{24}$ cm sec^{-2}

(*i*) $K = \dfrac{2\pi^2 me^4}{h^2} = \dfrac{mc^2}{2}\dfrac{e^4}{\hbar^2 c^2} = \dfrac{0.511 \times 10^6}{2}\left(\dfrac{1}{137}\right)^2$ ev $= 13.6$ ev.

(*j*) $U = -\dfrac{4\pi^2 me^4}{h^2} = -\dfrac{mc^2 e^4}{\hbar^2 c^2} = -0.511 \times \dfrac{10^6}{(137)^2} = -27.2$ ev.

(*k*) $E = -\dfrac{2\pi^2 me^4}{h^2} = -13.6$ ev.

Aliter: (b) $r=\dfrac{h^2\varepsilon_0}{\pi m e^2}$

$$=\frac{(6.63\times10^{-34}\text{ joule-sec})^2(8.85\times10^{-12}\text{ coul}^2/\text{nt-m}^2)}{\pi(9.1\times10^{-31}\text{ kg})(1.6\times10^{-19}\text{ coul})^2}$$

$=5.3\times10^{-11}$ meter.

(c) $L=\dfrac{h}{2\pi}=\dfrac{6.63\times10^{-34}\text{ joule-sec}}{2\pi}=1.06\times10^{-34}$ joule-sec.

(f) $$v=\sqrt{\frac{e^2}{4\pi\varepsilon_0 mr}}$$

$$=\sqrt{\frac{(1.6\times10^{-19}\text{ coul})^2(9\times10^9\text{ nt-m}^2/\text{coul}^2)}{(9.1\times10^{-31}\text{ kg})(5.3\times10^{-11}\text{ meter})}}$$

$=2.19\times10^6$ meter/sec.

(d) $p=mv=(9.1\times10^{-31}\text{ kg})(2.19\times10^6\text{ meter/sec})$

$=2\times10^{-24}$ kg-meter/sec.

(e) $\omega=\dfrac{v}{r}=\dfrac{2.19\times10^6\text{ meter/sec}}{5.3\times10^{-11}\text{ meter}}=4.1\times10^{16}$ radians/sec.

(g) $F=\dfrac{e^2}{4\pi\varepsilon_0 r^2}$

$$=\frac{(9\times10^9\text{ nt-m}^2/\text{coul}^2)(1.6\times10^{-19}\text{ coul})^2}{(5.3\times10^{-11}\text{ meter})^2}=8.2\times10^{-8}\text{ nt.}$$

(h) $a=\dfrac{F}{m}=\dfrac{8.2\times10^{-8}\text{ nt}}{9.1\times10^{-31}\text{ kg}}=9\times10^{22}$ meter/sec^2.

(i) $K=\dfrac{e^2}{8\pi\varepsilon_0 r}$

$=\frac{1}{2}(9\times10^9\text{ nt-m}^2/\text{coul}^2)(1.6\times10^{-19}\text{ coul})^2/(5.3\times10^{-11}\text{ m})$

$=2.17\times10^{-18}$ joules

$=(2.17\times10^{-18}\text{ joule})/(1.6\times10^{-19}\text{ joule/ev})=13.6$ ev.

(j) $U=-\dfrac{e^2}{4\pi\varepsilon_0 r}=-2\text{ K}=-27.2$ ev.

(k) $E=K+U=(13.6-27.2)\text{ ev}=-13.6$ ev.

47.18. (b) $r\propto n^2$ (c) $L\propto n$ (d) $p\propto\dfrac{1}{n}$

(e) $\omega\propto\dfrac{1}{n^3}$ (f) $v\propto\dfrac{1}{n}$ (g) $F\propto\dfrac{1}{n^4}$

(*h*) $\propto_a \frac{1}{n^4}$ (*i*) $K \propto \frac{1}{n^2}$ (*j*) $U \propto \frac{1}{n^3}$ (*k*) $E \propto \frac{1}{n^2}$

47.19. As the energy required to remove the electron is $\propto (1/n^2)$ and since ionization potential (i.e. energy required to remove the electron from the ground state of hydrogen atom with $n=1$) is 13.6 ev, energy required to remove the electron from the $n=8$ state is

$$\frac{13.6}{8^2} \quad \text{or} \quad 0.21 \text{ ev.}$$

47.20. $$h\nu = \frac{2\pi^2 m e^4 h}{h^3}\left(\frac{1}{j^2}-\frac{1}{k^2}\right)$$

$$=(3.28\times 10^{15} \text{ sec}^{-1})(4.14\times 10^{-15} \text{ ev sec})\left(\frac{1}{j^2}-\frac{1}{k^2}\right)$$

$$=13.6\left(\frac{1}{j^2}-\frac{1}{k^2}\right)$$

(*a*) For $j=1$ and $k=4$

$$E=h\nu=13.6\left(\frac{1}{1^2}-\frac{1}{4^2}\right)=12.75 \text{ ev.}$$

(*b*) For transition from $n=4$ to $n=3$, $E=0.66$ ev. For transition from $n=4$ to $n=2$, $E=2.55$ ev. For transition from $n=4$ to $n=1$, $E=12.75$ ev.

For transition from $n=3$ to $n=2$, $E=1.89$ ev.

For transition from $n=3$ *to* $n=1$, $E=12.09$ ev.

For transition from $n=2$ to $n=1$, $E=10.2$ ev.

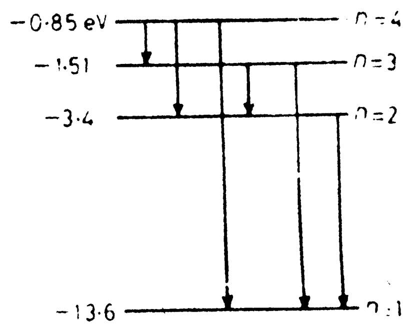


Fig. 47.20

(*c*) Transition energy, $E_0=12.75$ ev.

This energy must be shared between the emitted photon of energy $h\nu$ and recoiling hydrogen atom,

$$E_R + h\nu = 12.75 \text{ ev} \qquad \ldots(1)$$

Momentum conservation gives

$$P_R = h\nu/c$$

i.e. $$\sqrt{2E_R M_H} = \frac{h\nu}{c}$$

or $\qquad h\nu=\sqrt{2E_R M_H c^2}$ ...(2)

Using the value of $h\nu$ from (2) in (1)

$$E_R+\sqrt{2\,E_R M_H c^2}=12.75$$

$$\tfrac{1}{2}M_H v_H^2+\sqrt{2\times\tfrac{1}{2}M_H v_H^2 M_H c^2}=12.75$$

or $$v_H^2+2\,v_H c=\frac{25.5}{M_H}$$

$$\frac{v_H^2}{c^2}+\frac{2v_H}{c}=\frac{25.5}{M_H c^2}=\frac{25.5 \text{ ev}}{940\times10^6 \text{ ev}}$$

$$\beta^2+2\,\beta-2.7\times10^{-8}=0$$

where $\qquad \beta=v_H/c$

Solution of the quadratic equation gives,

$$\beta=1.35\times10^{-8}$$

$\therefore \qquad v_H=\beta c=(1.35\times10^{-8})(3\times10^{10}\text{ cm/sec})=405\text{ cm/s.}$

47.21. (*a*) $h\nu=-0.85-(-3.4)=2.55$ ev

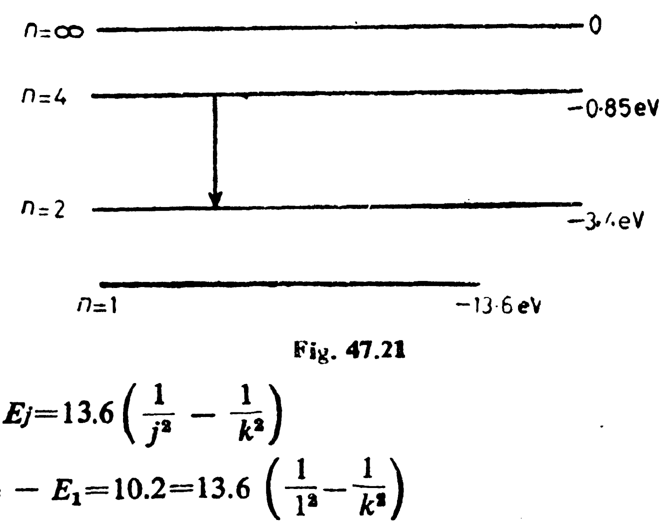


Fig. 47.21

(*b*) $E_k-E_j=13.6\left(\frac{1}{j^2}-\frac{1}{k^2}\right)$

With $E_k-E_1=10.2=13.6\left(\frac{1}{1^2}-\frac{1}{k^2}\right)$

or $\qquad k=\sqrt{\frac{13.6}{3.4}}=2$

With $E_m-E_1=-0.85-(-13.6)=13.6\left(\frac{1}{1^2}-\frac{1}{m^2}\right)$

we find $\qquad m=4.$

47.22. $\lambda=1216$ A°

$$h\nu=\frac{hc}{\lambda}=\frac{(4.14\times10^{-15}\text{ ev})(3\times10^{10}\text{ cm/sec})}{(1216\times10^{-8}\text{ cm})}=10.21 \text{ ev}$$

$$10.21 = 13.6\left(\frac{1}{j^2} - \frac{1}{k^2}\right)$$

Assigning different integral values for j and k, we find that the above equation is satisfied for $j=1$ and $k=2$. The energy level diagram is shown in Fig. 47.23.

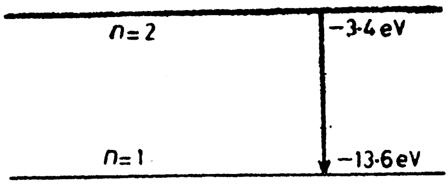


Fig 47.23

47.23. Since the excitation energy for the state $n=2$ is 10.2 ev, the kinetic energy of neutron (6.0 ev) falls short of it. Therefore, the collision can be only elastic one. The initial energy of 6.0 ev will be shared between the scattered neutron and the recoiling hydrogen atom.

47.24. For singly ionized helium, the nuclear charge will be $+2e$. The expression for photon energy will be multiplied by a factor of 2^2 *i.e.* 4. The corresponding spectrum (apart from correcting for reduced mass) will be pushed up on the frequency scale by a factor of 4 compared to hydrogen spectrum (or λ shortened by a factor of 4).

47.25. $h\nu = (4)(13.6 \text{ ev}) = 54.4$ ev.

47.26. In Bohr's formulae m should be replaced by the reduced mass μ.

$$\frac{1}{\mu} = \frac{1}{m} + \frac{1}{m}, \quad \text{or} \quad \mu = \frac{m}{2}$$

(*a*) $$h\nu = \frac{2\pi^2 \mu e^4}{h^2}\left(\frac{1}{j^2} - \frac{1}{k^2}\right) = \frac{\pi^2 m e^4}{h^2}\left(\frac{1}{j^2} - \frac{1}{k^2}\right)$$

Compared to hydrogen spectrum, it is seen that the frequencies in positronium spectrum are shrunk by a factor of 2, (or λ increased by a factor of 2).

(*b*) $$r_{(Pos)} = \frac{h^2}{4\pi^2 \mu e^2} = \frac{2h^2}{4\pi^2 m e^2} = 2r_H$$

where r_H = ground state radius of hydrogen atom = 0.53 A°.

$\therefore \quad r_{(Ps)} = 2 \times 0.53 = 1.06$ A.

47.27. (*a*) $$r_{(Muon)} = \frac{h^2}{4\pi^2 \mu e^2}$$

where μ is the reduced mass of μ and proton system.

$$\frac{1}{\mu} = \frac{1}{1836 \text{ m}} = \frac{1}{267 \text{ m}}$$

where m is the mass of electron.

$\therefore \qquad \mu \doteq 186m$

$$r_{(Muon)}=\frac{h^2}{186\times 4\pi^2 me^2}=\frac{r_H}{186}=\frac{0.53\times 10^{-8}}{186}\text{ cm}$$

$$=0.285\times 10^{-10}\text{ cm.}$$

(*b*) As the reduced mass is 186 times greater, the ionization energy is enhanced by the same factor; i.e. ionization energy for muonic atom with proton as nucleus is

(13.6 ev)(186) or 2530 ev, i.e. 2.53 kev.

(*c*) Maximum transition energy is provided for transition between $n=2$ and $n=1$. For muonic atom,

$$h\nu=2530\left(\frac{1}{1^2}-\frac{1}{2^2}\right)=1897\text{ ev}$$

$$\nu=\frac{1897}{4.14\times 10^{-15}}\text{ sec}^{-1}$$

$$\lambda=\frac{c}{\nu}=\frac{(3\times 10^{10}\text{ cm/sec})(4.14\times 10^{-15}\text{ ev-sec})}{(1897\text{ ev})}$$

$$=6.5\times 10^{-8}\text{ cm.}$$

a wavelength which falls in the X-ray region.

48.28. Equating the electrical force to centripetal force,

$$\frac{e^2}{r^2}=m\omega^2 r \qquad \text{.. (1)}$$

Quantization of angular momentum gives

$$m\omega r^2=\frac{nh}{2\pi} \qquad \text{...(2)}$$

Squaring (1) and re-arranging,

$$m^2\omega^4 r^6=e^4 \qquad \text{...(3)}$$

Raising both the sides of (2) to power 3,

$$m^3\omega^3 r^6=\frac{n^3h^3}{8\pi^3} \qquad \text{...(4)}$$

Divide (3) by (4) to get

$$\nu_0=\frac{\omega}{2\pi}=\frac{4\pi^2 me^4}{n^3h^3} \qquad \text{...(5)}$$

The frequency of radiation is given by

$$\nu=\frac{2\pi^2 me^4}{h^3}\left(\frac{1}{j^2}-\frac{1}{k^2}\right)$$

for transitions between the states k and j.

Setting $j=n$ and $k=n+1$

$$\nu=\frac{2\pi^2me^4}{h^3}\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)$$

$$=\frac{2\pi^2me^4\,(2n+1)}{h^3n^2(n+1)^2} \qquad \ldots\ldots(6)$$

If n is large then in the parenthesis of both numerator and denominator of equation (6), 1 can be neglected so that

$$\frac{(2n+1)}{(n+1)^2}\longrightarrow\frac{2n}{n^2}=\frac{2}{n}$$

and $$\nu\longrightarrow\frac{4\pi^2me^4}{h^3n^3}=\nu_0$$

Next letting $j=n$ and $k=n+2$

$$\nu=\frac{2\pi^2me^4}{h^3}\left[\frac{1}{n^2}-\frac{1}{(n+2)^2}\right]=\frac{8\pi^2me^4\,(n+1)}{h^3n^2\,(n+2)^2}$$

For large values of n,

$$\frac{(n+1)}{(n+1)^2}\longrightarrow\frac{n}{n^2}=\frac{1}{n}$$

$$\nu\longrightarrow\frac{8\pi^2me^4}{n^3h^3}=2\nu_0$$

We may repeat the calculation for transition between states with quantum numbers $n+3$ and n to find $\nu=3\nu_0$. Thus, in the limit of large quantum numbers Bohr's theory predicts $\nu=\nu_0$, $2\nu_0$, $3\nu_0$ etc.

47.29. For $k=n$, $j=n-1$, frequency of emitted radiation is given by

$$\nu=\frac{2\pi^2me^4}{h^3}\left[\frac{1}{(n-1)^2}-\frac{1}{n^2}\right]$$

$$=\frac{2\pi^2me^4}{h^3}\left[\frac{2n-1}{(n-1)^2\,n^2}\right]$$

The orbital frequency ν_0 is given by

$$\nu_0=\frac{4\pi^2me^4}{n^3h^3}$$

$\therefore$ Percentage difference is $100\left(\frac{\nu-\nu_0}{\nu}\right)$

$$=\frac{100\left[\frac{2n-1}{(n-1)^2\,n^2}-\frac{2}{n^3}\right]}{\frac{2n-1}{(n-1)^2\,n^2}}=\frac{100\,(3n-2)}{n\,(2n-1)}$$

For large n, $3n-2\longrightarrow 3n$ and $2n-1\longrightarrow 2n$

$\therefore$ Percentage difference$=100\times\dfrac{3n}{n\,(2n)}=\dfrac{150}{n}$.

47.30. $L=mvr=\dfrac{nh}{2\pi}$

$$n=\frac{2\pi mvr}{h}$$

Mass of earth, $m=6\times10^{24}$ kg.

Mean orbital speed of earth, $v=29770$ meters/sec.

Planck's constant $h=6.63\times10^{-34}$ joules-sec.

Orbital radius, $r=1.5\times10^{11}$ meters.

$$n=\frac{2\pi\times6\times10^{24}\times2.977\times10^{4}\times1.5\times10^{11}}{6.63\times10^{-34}}$$

$$n=254\times10^{72}=2.5\times10^{74}$$

Such a quantization can not be detected.

48 WAVES AND PARTICLES

48.1. (*a*) deBroglie wavelength, $\lambda = \dfrac{h}{p} = \dfrac{h}{mv}$

$$\lambda = (6.6 \times 10^{-34}\ \text{joules-sec})/(0.04\ \text{kg} \times 1000\ \text{meter/sec})$$
$$= 1.65 \times 10^{-35}\ \text{meter}$$

(*b*) Diffraction effects are noticeable for obstacles which have dimensions of the order of wavelength. But, here λ is too small. Thus, for obstacles of the size of atoms ($R \sim 10^{-10}$ meters) the diffraction angle $\theta \sim \dfrac{\lambda}{R} = \dfrac{10^{-35}\ \text{meters}}{10^{-10}\ \text{meter}} = 10^{-25}$ radians an angle which is beyond detection.

48.2. deBroglie wavelength, $\lambda = \dfrac{h}{p} = \dfrac{h}{\sqrt{2m_n K}}$

$$\lambda = \frac{(6.6 \times 10^{-34}\ \text{joule-sec})}{\sqrt{2 \times (1.67 \times 10^{-27}\ \text{kg})(0.025\text{ev} \times 1.6 \times 10^{-9}\ \text{joule/ev})}}$$
$$= 1.84 \times 10^{-10}\ \text{meter} = 1.84\ \text{A}.$$

48.3. (*a*) Momentum of electron, $p = \dfrac{h}{\lambda} = \dfrac{6.6 \times 10^{-34}\ \text{joule-sec}}{2.0 \times 10^{-10}\ \text{meter}}$

$$= 3.3 \times 10^{-24}\ \text{kg-meter/sec}$$

Momentum of photon, $p = \dfrac{h\nu}{c} = \dfrac{h}{\lambda} = 3.3 \times 10^{-24}$ kg-meter/sec.

(*b*) For electron, kinetic energy, $K = \dfrac{p^2}{2m_e}$

$$K = (3.3 \times 10^{-24}\ \text{kg-meter/sec})^2/(2 \times 9.1 \times 10^{-31}\ \text{kg})$$
$$= 0.6 \times 10^{-17}\ \text{joules}$$
$$= (0.6 \times 10^{-17}\ \text{joules})/(1.6 \times 10^{-19}\ \text{joules/ev})$$
$$= 37.5\ \text{ev}.$$

For proton, energy, $E = h\nu = cp$

$$E = (3 \times 10^8\ \text{meter/sec})(3.3 \times 10^{-24}\ \text{kg-meter/sec})$$
$$= 9.9 \times 10^{-16}\ \text{joules}$$
$$= (9.9 \times 10^{-16}\ \text{joules})/(1.6 \times 10^{-19}\ \text{joules/ev})$$
$$= 6188\ \text{ev}$$

48.4. Kinetic energy of electron, $K=50$ Bev $=5\times10^{10}$ ev

So $K \gg m_0c^2$; the rest mass energy of electron which is only 0.51×10^6 ev.

Now, total energy, $E=K+m_0c^2$ is given by relativistic formula

$$E^2=c^2p^2+m^2{}_0c^4$$

$$\therefore \qquad (K+m_0c^2)=c^2p^2+m^2{}_0c^4$$

or $$K^2+2Km_0c^2=c^2p^2$$

If $K \gg m_0c^2$, the second term on the left-hand side will be much smaller than the first term,

$$\therefore \quad K^2\simeq c^2p^2$$

or $$p\simeq\frac{K}{c}=\frac{E}{c}$$

Momentum, $p=\dfrac{E}{c}$

$$=(5\times10^{10}\text{ ev}\times1.6\times10^{-19}\text{ joule/ev})/(3.0\times10^8\text{ meter/sec})$$
$$=2.7\times10^{-17}\text{ kg-meter/sec.}$$

deBroglie wavelength, $\lambda=\dfrac{h}{p}$

$$=6.6\times10^{-34}\text{ joules})/(2.7\times10^{-17}\text{ kg-meter/sec})$$
$$=2.45\times10^{-17}\text{ meter.}$$

On the basis of constant density nuclear model, nuclear radius,

$$R=r_0A^{1/3}=1.2\times10^{-15}A^{1/3}\text{ meter,}$$

where A=mass number.

For a typical medium size nucleus with $A=125$, wh get

$$R=6.0\times10^{-15}\text{ meter.}$$

Thus λ is comparable with R and diffraction effects will be prominent.

48.5. (*a*) deBroglie wavelength of electrons of kinetic energy $T=54$ ev$=(54\text{ev})\times(1.6\times10^{-19}$ joules/ev$)=86.4\times10^{-11}$ joules is

$$\lambda=\frac{h}{\sqrt{2mK}}=\frac{6.6\times10^{-34}\text{ joule-sec}}{\sqrt{(2)(9.1\times10^{-31}\text{ kg})(86.4\times10^{-19}\text{ joules})}}$$
$$=1.65\times10^{-10}\text{ meter}$$

Now, condition for Bragg-reflection is

$$m\lambda=2d\sin\theta$$

Set $m=2$ (second order diffraction)

then, $\sin\theta=\frac{2\lambda}{2d}=\frac{1.65\times10^{-10}\text{ meter}}{0.91\times10^{-10}\text{ meter}}=1.813$

which is impossible since the value of sin θ can not exceed unity. Hence, second order diffraction cannot occur. Similarly, third order diffraction also cannot occur with the given accelerating voltage (which defines λ) and the set of planes (which define d).

(*b*) Set $V=60$ volts

Then, kinetic energy of electrons,

$$K=60\text{ ev}=(60\text{ ev})(1.6\times10^{-19}\text{ joule/ev})=9.6\times10^{-18}\text{ joule.}$$

Momentun, $p=\sqrt{2m_eK}=\sqrt{(2)(9.1\times10^{-31}\text{ kg})(9.6\times10^{-18}\text{ joules})}$

$$=4.18\times10^{-24}\text{ kg-meter/sec.}$$

$$\lambda=\frac{h}{p}=(6.6\times10^{-34}\text{ joule-sec})/(4.18\times10^{-24}\text{ kg-meter/sec})$$

$$=1.59\times10^{-10}\text{ meter}$$

For first order diffraction, $\lambda=2d\sin\theta$

or $\sin\theta=\frac{\lambda}{2d}=(1.59\times10^{-10}\text{ meter})/(2\times0.91\times10^{-10}\text{ meter})$

$$=0.874$$

∴ Bragg angle, $\theta=61°$.

But, $\theta=90-\frac{1}{2}\phi$

whence $\phi=180-2\theta=180°-2\times61°=58°$.

48.6. (*a*)

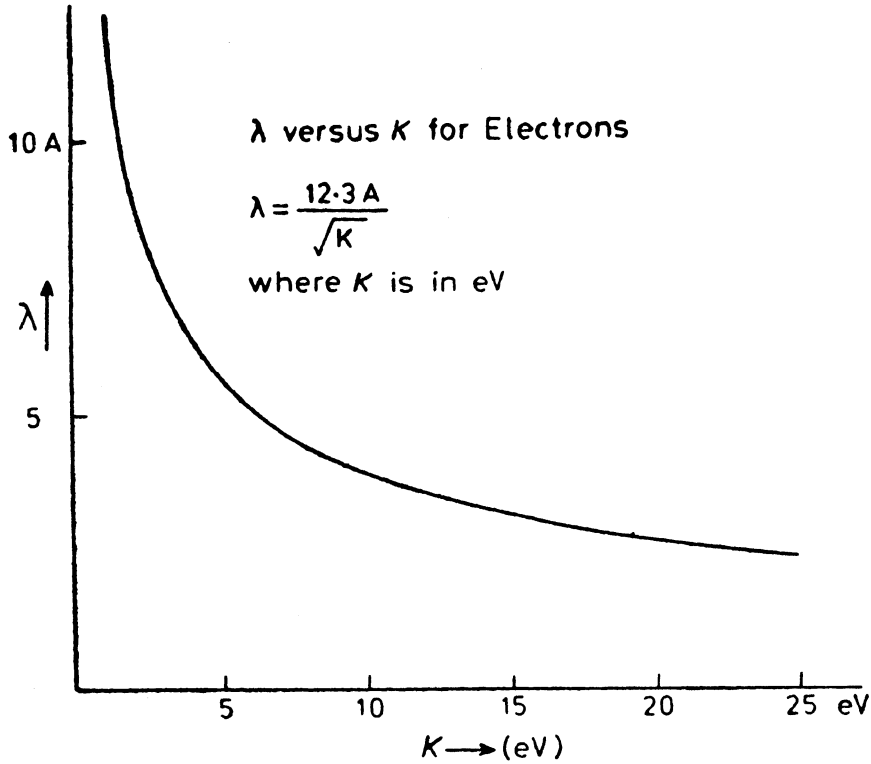


Fig 48.6 (a)

(*b*)

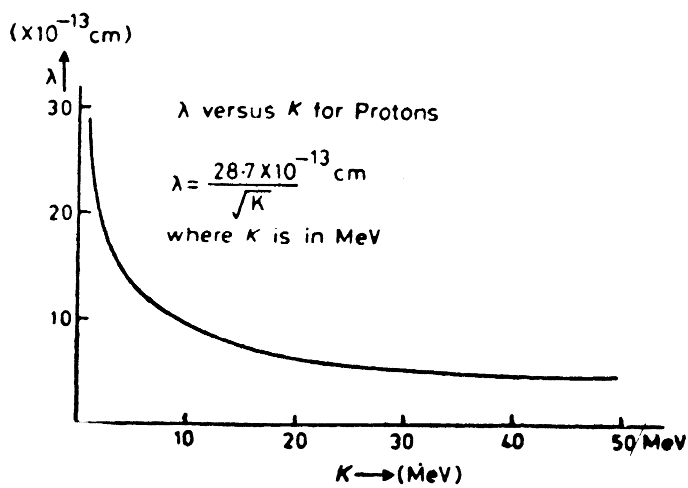


Fig. 48.6 (b)

Mean kinetic energy of hydrogen atom at temperature T vin) is

$$\bar{E}=\frac{3}{2}\,kT, \text{ where } k=\text{Boltzmann's constant}$$

$$\text{constant}=1.38\times10^{-23}\text{ joule/K}^\circ$$

$$T=273^\circ+20^\circ=293^\circ\text{K}$$

$$\bar{E}=\left(\frac{3}{2}\right)(1.38\times10^{-23}\text{ joule/}^\circ\text{K})(293^\circ\text{K})=6.0\times10^{-21}\text{ joules}$$

$$m_H=1.67\times10^{-27}\text{ kg}$$

$$\lambda=\frac{h}{\sqrt{2\,\bar{E}\,m_H}}=\frac{6.63\times10^{-34}\text{ joule-sec}}{\sqrt{(2)(6\times10^{-21}\text{ joule})(1.67\times10^{-27}\text{ kg})}}$$

$$=1.48\times10^{-10}\text{ meter}=1.48\text{ A.}$$

48.8. (*a*) $h\nu=E_3-E_2=3.4-1.5=1.9$ ev.

(*b*) Energy of photon, $E=1.9\text{ ev}=(1.9\text{ ev})(1.6\times10^{-19}\text{ joules/ev})$

$$=3\times10^{-19}\text{ joules}$$

$$E=h\nu=\frac{hc}{\lambda},$$

$$\therefore\quad \lambda=\frac{hc}{E}$$

$$=\frac{(6.63\times10^{-34}\text{ joule-sec})(3\times10^{8}\text{ meter/sec})}{3\times10^{-19}\text{ joule}}$$

$$=6.63\times10^{-7}\text{ meter}=6630\text{ A.}$$

48.9. The normalized, time independent wave function for the particle trapped in an infinitely deep potential well of width L is

$$=\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$$

Probability of finding the particle between x and $x+dx$ in state n, is

$$P(x)dx=|\psi|^2\,dx=\frac{2}{L}\sin^2\frac{n\pi x}{L}\,dx$$

Probability of finding the particle between $x=L/3$ and $x=0$, is

$$P=\int_0^{L/3}|\psi|^2\,dx=\int_0^{L/3}\frac{2}{L}\sin^2\frac{n\pi x}{L}\,dx$$

$$=\frac{2}{L}\int_0^{L/3}\left[\frac{1-(\cos 2n\pi x/L)}{2}\right]dx$$

$$=\frac{1}{L}\int_0^{L/3} dx-\frac{1}{L}\int_0^{L/3}\cos\frac{2n\pi x}{L}\,dx$$

$$=\frac{1}{3}-\frac{1}{L}\,\frac{L}{2\,n\pi}\left[\sin\frac{2n\pi x}{L}\right]_0^{L/3}$$

or
$$P=\frac{1}{3}-\frac{1}{2n\pi}\sin\frac{2n\pi}{3}$$

(*a*) $P_{(n=1)}=\frac{1}{3}-\frac{1}{2\pi}\sin\frac{2}{3}\pi=\frac{1}{3}-\frac{1}{2\pi}\times 0.866=0.19$

(*b*) $P_{(n=2)}=\frac{1}{3}-\frac{1}{4\pi}\sin\frac{4\pi}{3}=0.33+\frac{1}{4\pi}\times 0.866=0.4$

(*c*) $P_{(n=3)}=\frac{1}{3}-\frac{1}{6\pi}\sin 2\pi=0.33.$

(*d*) Classically, probability for the particle between x and $x+dx$ is dx/L.

$$\therefore\qquad P=\int_0^{L/3}\frac{dx}{L}=\frac{1}{3}=0.33.$$

48 10. Time independent normalized wave function for hydrogen atom in the ground state is $\psi_0=\frac{1}{\sqrt{\pi a^3}}e^{-r/a}$.

$$P_r=\int_0^r|\psi_0|^2\,dV=\frac{1}{\pi a^3}\int_0^r e^{-2r/a}(4\pi r^2dr)$$

$$=\frac{4}{a^3}\int_0^r r^2e^{-2r/a}\,dr\cdot$$

Set $\frac{2r}{a}=x\,;\ dr=\frac{a}{2}dx$

$$P_r=\left(\frac{4}{a^3}\right)\left(\frac{a^2}{4}\right)\left(\frac{a}{2}\right)\int x^2e^{-x}dx.$$

Integrate by parts,

$$P_r=\frac{1}{2}\left[-x^2e^{-x}+2\int xe^{-x}dx\right]$$

$$=\frac{1}{2}\left[-x^2e^{-x}+2\left\{-xe^{-x}+\int e^{-x}\right\}\right]$$

$$=\frac{1}{2}\left[-x^2e^{-x}-2xe^{-x}-2e^{-x}\right]$$